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New family of exact stationary axisymmetric gravitational fields generalising the Tomimatsu–Sato solutions

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Abstract. A new three-parameter family of exact solutions of the stationary axisymmetric vacuum Einstein equations, which represent rotating bounded sources, are presented. This family contains the solutions of Kerr and Tomimatsu–Sato as special cases, and may be regarded as a generalisation of the latter to arbitrary continuous δ parameter. The final form of the metric depends on two ordinary differential equations of the second order. When δ is not an integer, these equations define unfamiliar transcendental functions for which rapidly converging series expansions of several types are available. When δ is an integer, the solutions are polynomial or rational functions of spheroidal coordinates and define the discrete Tomimatsu–Sato series, for which those authors give the cases $\delta = 1, 2, 3, 4$. One of the two equations is solved explicitly for the case $\delta = 5$ and efficient algorithms are presented which make it possible to perform such calculations by hand. The metric and Ernst potentials assume simple functional forms on the symmetry axis. Actually, this three-parameter family of asymptotically flat solutions is shown to be contained in a family of unphysical solutions with six non-trivial parameters, one of which is the familiar NUT parameter.

1. Introduction

In Einstein's general theory of relativity, exact solutions of the field equations representing the vacuum exterior of bounded gravitational sources are still exceedingly scarce. The cases of static axisymmetry and non-static spherical symmetry have been completely solved (Weyl 1917, Birkhoff 1923, respectively). The next simplest vacuum gravitational fields are those with stationary axisymmetry. These are very important in relativistic astrophysics, being needed, for example, for neutron stars, quasars and dense star clusters, where strong fields and rapid rotation are expected. But in this case the field equations, though deceptively simple in form, are unfortunately very difficult to solve. Though large classes of unphysical solutions are known, the list of known astrophysically admissible solutions numbers exactly four, each containing two arbitrary parameters, m and q, representing the mass (m) and angular momentum (m^2q) of the source.

The first of these is the celebrated solution of Kerr (1963). (Other derivations: Newman and Janis (1965), Carter (1968a), Ernst (1968); for global properties, see Carter (1968b).) The importance of this solution as the vacuum exterior of a rotating star is far overshadowed by its interpretation as a black hole (see, e.g., Hawking and Ellis 1973). The next was found by Tomimatsu and Sato (1972, to be referred to as TS 1972) and was followed by two more (TS 1973). The TS solutions were obtained by using the Ernst complex potential formalism (Ernst 1968, 1974), in prolate spheroidal coordinates, and a computer to search for rational function solutions aided by a few rules of computation. The Kerr and TS solutions are labelled by a discrete parameter, δ, taking values $\delta = 1, 2, 3, 4$, which measures the mass quadrupole, Q, of the source according to

$$Q = m^3[q^2 + p^2(\delta^2 - 1)/3\delta^2], \tag{1.1}$$

where $p^2 = 1 - q^2$.

In this paper, we shall generalise the Kerr and TS solutions to allow the parameter δ to take any real or pure imaginary value. The final form of the metric depends on two independent ordinary differential equations (DE) of the second order satisfying certain boundary conditions. One of these is a linear Fuchsian equation with five regular singular points similar to Lamé's equation; the other is non-linear, but reasonably simple in form. There are distinct advantages, however, in working with alternative third- or fourth-order DE. When δ is an integer, these equations admit polynomial or rational function solutions which can be readily identified with the Kerr and TS solutions when $\delta = 1, 2, 3, 4$. When δ is not an integer, the equations define transcendental functions for which efficient series expansions are available. This three-parameter family of solutions can be expanded to a family of six non-trivial parameters by introducing a new parameter, h, to the two DE and by relaxing the boundary conditions. One of these additional parameters is the familiar NUT parameter (see § 2). Since these additional solutions are unphysical, they will be only briefly outlined.

These new gravitational solutions are not derived from first principles in this paper since such a derivation would make this paper prohibitively long. The full derivation is given in Cosgrove (1977e). These solutions (the full six-parameter family plus a few trivial parameters) are there shown to arise from a transformation group, also new, which transforms any stationary axisymmetric vacuum solution into another such solution whilst preserving the boundary conditions of asymptotic flatness, regular symmetry axis and bounded singularities. All previously known transformation groups create a line singularity on half or all of the symmetry axis and destroy asymptotic flatness (in the sense of weak asymptotic simplicity—see Hawking and Ellis 1973). (For examples of such groups or solutions resulting from their application, see Papapetrou (1953), Ehlers (1962, 1965), Ernst (1968, 1974), Geroch (1971, 1972), Lewis (1932), Matzner and Misner (1967), Kinnersley (1973).)

This paper is set out as follows. The two basic ordinary differential equations are written down in § 3 and explicit formulae for the metric and Ernst potentials are given in § 4. Methods of solving the ordinary DE are postponed till § 10. In §§ 5 and 6, we demonstrate that Einstein's vacuum field equations are satisfied and so also is 'rule (a)' of Tomimatsu and Sato (1973). In §§ 7 and 8, we prove that the solutions are asymptotically flat and are well behaved on the symmetry axis and very simple formulae are given for the metric coefficients and Ernst potentials on the symmetry axis in § 9. Finally, in § 11, we give an efficient algorithm for calculating the cases when δ is an integer and these results may be directly compared with the Kerr and TS solutions when $\delta = 1, 2, 3, 4$.

The metric coefficients are analytic functions of the parameters δ^{-2} and q for $-\infty < \delta^{-2} < \infty$ and $-\infty < q < \infty$, when viewed in canonical cylindrical coordinates, outside the inner regions, presumed covered by the material source of the gravitational field, where the analytically extended vacuum solution may be singular. However, we shall concentrate attention on the ranges $0 < \delta < \infty$, $-1 < q < 1$, though it will be clear how our equations may be adapted to the cases $|q| > 1$ and/or δ finite pure imaginary.

The limiting cases, $\delta = \infty$ and $q = \pm 1$, will be treated in a separate paper (Cosgrove 1977c). These correspond to a 'rotating Curzon' metric and a generalised 'extreme Kerr' metric, respectively. A further paper (Cosgrove 1977d) will present a formulation of the stationary axisymmetric gravitational field equations alternative to those of Einstein (as arranged by Lewis 1932) and Ernst (1968, 1974) from which the present family of new solutions and another similar but unphysical class arise quite naturally.

2. Einstein's and Ernst's equations

2.1. Field equations

Take the metric of space–time in the Weyl–Lewis–Papapetrou canonical form (Lewis 1932):

$$ds^2 = e^{2u}(dt - \omega\, d\phi)^2 - e^{-2u}[e^{2\gamma}(dr^2 + dz^2) + r^2\, d\phi^2], \tag{2.1}$$

where r, ϕ, z are cylindrical coordinates, t is time and u, ω, γ are functions of r and z only. Einstein's vacuum field equations reduce to

$$u_{rr} + u_{zz} + (1/r)u_r + (1/2r^2)\, e^{4u}(\omega_r^2 + \omega_z^2) = 0, \tag{2.2a}$$

$$\omega_{rr} + \omega_{zz} - (1/r)\omega_r + 4(u_r\omega_r + u_z\omega_z) = 0, \tag{2.2b}$$

where $u_r \equiv \partial u/\partial r$, $u_{rr} \equiv \partial^2 u/\partial r^2$, etc. Ehlers and Kundt (1962), Ehlers (1965) and Ernst (1968) chose a new potential ψ according to

$$\psi_r = (1/r)\, e^{4u}\omega_z, \qquad \psi_z = -(1/r)\, e^{4u}\omega_r. \tag{2.3}$$

Compatibility of the two equations (2.3) is guaranteed by the second Einstein equation, (2.2b). The field equations can be rewritten:

$$u_{rr} + u_{zz} + (1/r)u_r + \tfrac{1}{2}\, e^{-4u}(\psi_r^2 + \psi_z^2) = 0, \tag{2.4a}$$

$$\psi_{rr} + \psi_{zz} + (1/r)\psi_r - 4(u_r\psi_r + u_z\psi_z) = 0. \tag{2.4b}$$

These are Ernst's equations. Ernst (1968) then chose two complex potentials,

$$\mathscr{E} = e^{2u} + i\psi, \tag{2.5a}$$

$$\xi = (1 + \mathscr{E})/(1 - \mathscr{E}), \tag{2.5b}$$

each of which satisfies a single compact and elegant field equation, but which does not uncouple equations (2.4) because of the appearance of the complex conjugates, $\mathscr{E}^*$ and ξ^*. The complex Ernst potential ξ assumes a simple form for many known solutions, especially the Kerr solution, and has been very useful in finding new ones. However, surprisingly, $\mathscr{E}$ and ξ do not arise naturally in our work and the original derivation from first principles proceeded, of necessity, in the (u, ω) formulation. The metric coefficient $e^{2\gamma}$ can be obtained from

$$\gamma_r + i\gamma_z = r(u_r + iu_z)^2 - (1/4r)\, e^{4u}(\omega_r + i\omega_z)^2, \tag{2.6a}$$

$$= r[(u_r + iu_z)^2 + \tfrac{1}{4}\, e^{-4u}(\psi_r + i\psi_z)^2], \tag{2.6b}$$

compatibility being guaranteed by the field equations, (2.2) or (2.4).

Alternative coordinate systems will be needed. The spherical-like coordinates (ρ, θ) (not to be confused with Schwarzschild coordinates), defined by

$$r = \rho \sin\theta, \qquad z = \rho\cos\theta, \tag{2.7}$$

will be useful in discussing the asymptotically flat outer regions. The much more important prolate spheroidal coordinates (x, y) are defined by

$$r = \kappa(x^2-1)^{1/2}(1-y^2)^{1/2} \qquad z = \kappa x y, \tag{2.8}$$

where κ is a positive constant. x is a radial coordinate, y angular with range $-1 \leqslant y \leqslant 1$. Many known solutions, including the Kerr and TS solutions, take relatively simple forms in these coordinates for appropriate κ.

Remarkably, the system of coordinates most strongly preferred for our solutions is not prolate spheroidal but a rather unusual system:

$$\nu = y/x, \qquad \eta = (x^2-1)/(1-y^2). \tag{2.9}$$

These coordinates, which are orthogonal and depend on κ, are rather poorly suited to the asymptotically flat outer regions and break down on the symmetry axis, $y^2 = 1$. Nevertheless, these are the independent variables in the two ordinary differential equations which determine our solutions. Note that large η corresponds both to the distant outer regions and the neighbourhood of the symmetry axis.

2.2. *Some well known solutions*

Let a solution of (2.2) or (2.4) represent an astrophysically meaningful rotating body with mass $m > 0$, angular momentum J, vanishing mass dipole and mass quadrupole Q, in geometrical units. Then in (ρ, θ) coordinates, the functions u, ω and ψ adopt the following asymptotic behaviour:

$$u = -m\rho^{-1} + (Q - \tfrac{1}{3}m^3)(\tfrac{3}{2}\cos^2\theta - \tfrac{1}{2})\rho^{-3} + O(\rho^{-4}), \tag{2.10a}$$

$$\omega = -(2J\sin^2\theta)\rho^{-1} + O(\rho^{-2}), \tag{2.10b}$$

$$\psi = -(2J\cos\theta)\rho^{-2} + O(\rho^{-3}), \tag{2.10c}$$

as $\rho \to \infty$. The coefficients of higher powers of ρ^{-1} will involve higher multipoles of mass and angular momentum type. These formulae will be regarded as boundary conditions at infinity for the partial differential equations, (2.2) and (2.4). Other boundary conditions, which will ensure a well behaved symmetry axis, are that the coefficients of the power series in ρ^{-1} for u, $(\operatorname{cosec}^2\theta)\omega$ and ψ be polynomials in $\cos\theta$. Similarly, the coefficients of power series in x^{-1} for u, $(1-y^2)^{-1}\omega$ and ψ should be polynomials in y.

The static or non-rotating case, $\omega = 0$, was solved completely by Weyl (1917). Putting $\omega = 0$ in (2.2) yields

$$u_{rr} + u_{zz} + u_r/r = 0 \tag{2.11}$$

which is the axisymmetric Laplace equation in cylindrical coordinates. The general astrophysical solution involves one arbitrary function of one argument. Special solutions of interest are the Schwarzschild solution,

$$u = \frac{1}{2}\ln\left(\frac{x-1}{x+1}\right) \qquad \text{or, equivalently,} \qquad \xi = x, \tag{2.12}$$

with $\kappa = m$, and the Zipoy–Voorhees solutions (Zipoy 1966, Voorhees 1970),

$$u = \frac{1}{2}\delta \ln\left(\frac{x-1}{x+1}\right), \tag{2.13}$$

with $\kappa = m\delta^{-1}$. These fields have mass m and quadrupole $Q = m^3\delta^{-2}(\delta^2 - 1)$. δ may take pure imaginary values by transforming to oblate spheroidal coordinates $(\bar{x}, y)$ according to

$$x = \mathrm{i}\bar{x}, \qquad y = y, \qquad \kappa = \mathrm{i}\bar{\kappa}, \qquad \delta = -\mathrm{i}\bar{\delta}.$$

Also, the limit $\delta \to \infty$ (or $\bar{\delta} \to \infty$) is quite regular when viewed in (ρ, θ) coordinates. This is the Curzon metric,

$$u = -m/\rho. \tag{2.14}$$

In fact, u is an analytic function of δ^{-2} for $-\infty < \delta^{-2} < \infty$ with ρ, θ held fixed, ρ sufficiently large. Voorhees (1970) interpreted these solutions as the exterior gravitational fields of prolate spheroids for $1 < \delta^{-2} < \infty$, oblate spheroids for $-\infty < \delta^{-2} < 1$, and a sphere for $\delta = 1$.

The Kerr metric has the simple formula,

$$\xi = px - \mathrm{i}qy, \tag{2.15}$$

with $p^2 + q^2 = 1$, $\kappa = mp$ (Ernst 1968). This field has mass m, angular momentum $J = m^2 q$ and quadrupole $Q = m^3 q^2$. The potentials u, ω, ψ, γ are given by

$$\mathrm{e}^{2u} = \frac{p^2x^2 + q^2y^2 - 1}{(px+1)^2 + q^2y^2}, \tag{2.16a}$$

$$\omega = -\frac{2mq(1-y^2)(px+1)}{p^2x^2 + q^2y^2 - 1}, \tag{2.16b}$$

$$\psi = -\frac{2qy}{(px+1)^2 + q^2y^2}, \tag{2.16c}$$

$$\mathrm{e}^{2\gamma} = \frac{p^2x^2 + q^2y^2 - 1}{p^2(x^2 - y^2)}. \tag{2.16d}$$

This is the case $\delta = 1$ of the Tomimatsu–Sato series. The next case $\delta = 2$ (TS 1972, 1973) is given by

$$\xi = \frac{p^2x^4 + q^2y^4 - 1 - 2\mathrm{i}pqxy(x^2 - y^2)}{2px(x^2-1) - 2\mathrm{i}qy(1-y^2)}, \tag{2.17}$$

with $p^2 + q^2 = 1$, $\kappa = \frac{1}{2}mp$, $J = m^2 q$, $Q = \frac{1}{4}m^3(1 + 3q^2)$. The paper, TS (1973), then gives ξ for $\delta = 3, 4$ and u, ω, γ for $\delta = 1, 2, 3$. These latter functions are rational in x and y, if $\kappa = mp\delta^{-1}$, but the degree of the polynomial numerators and denominators are large, like $2\delta^2$ or $2\delta^2 + 1$. TS give seven 'rules for computation', labelled (a) to (g), for these polynomials and claim that they can also derive expressions for integers $\delta \geqslant 5$. These rules are rather weak, for purposes of practical computation by hand, since they leave a large number of coefficients undetermined, which can then only be determined by direct substitution into Ernst's ξ equation.

In this paper, we shall present a relatively quick algorithm for obtaining the solutions when δ is an integer (§ 11). A demonstration is given in the appendix where the metric coefficient $e^{2\gamma}$ is given for $\delta = 1, 2, 3, 4, 5$. However, the main objective is to generalise the TS solutions to continuous δ with $-\infty < \delta^{-2} < \infty$. These will be interpreted as the gravitational fields of rotating prolate or oblate spheroids as for the Zipoy–Voorhees metrics. When $q = 0$, these solutions reduce to the Zipoy–Voorhees metrics with the same m and δ.

We shall mainly concern ourselves with parameters in the ranges $0 < \delta < \infty$, $-1 < q < 1$. The extension to faster rotation, $|q| > 1$, is given by the substitution,

$$(x, y, \kappa, p, q, \delta) \to (\mathrm{i}\bar{x}, y, -\mathrm{i}\bar{\kappa}, -\mathrm{i}\bar{p}, q, \delta). \tag{2.18}$$

The extension to more flattening of the source (beyond $\delta = \infty$) is made by

$$(x, y, \kappa, p, q, \delta) \to (-\mathrm{i}\bar{x}, y, \mathrm{i}\bar{\kappa}, p, q, -\mathrm{i}\bar{\delta}). \tag{2.19}$$

Concerning limiting cases, see the last paragraph of § 1.

As noted by Ernst (1968), new solutions can be generated from old by the substitution,

$$\xi' = e^{\mathrm{i}\lambda}\xi, \tag{2.20}$$

where λ is a real constant, $0 \leq \lambda < 2\pi$. Except for $\lambda = 0$ and $\lambda = \pi$, this transformation creates a coordinate-type singularity on (at least) half of the symmetry axis, when the mass $m \neq 0$, and so does not generate astrophysical solutions. However, the parameter λ arises naturally in our solutions, so it is worthwhile to discuss this transformation briefly.

The parameter λ will be called the NUT parameter because of the well known result that the Schwarzschild metric transforms into the Taub–NUT metric (Newman *et al* 1963, Taub 1951; for properties, see Misner 1963, Bonnor 1969). Likewise, the Kerr metric transforms into the Kerr–NUT metric (Demianski and Newman 1966). The general static Weyl solutions transform into the Papapetrou–Ehlers solutions (Papapetrou 1953, Ehlers and Kundt 1962, Ehlers 1965). Each of these authors has a different method of derivation. Geroch (1971, 1972) has shown that this type of transformation exists for all vacuum gravitational fields admitting a Killing vector.

The explicit transformation equations for u, ψ and γ are

$$e^{2u'} = e^{2u}[(\cos\tfrac{1}{2}\lambda - \psi\sin\tfrac{1}{2}\lambda)^2 + e^{4u}\sin^2\tfrac{1}{2}\lambda]^{-1}, \tag{2.21a}$$

$$\psi' = -\cot\tfrac{1}{2}\lambda + (\cot\tfrac{1}{2}\lambda - \psi)[(\cos\tfrac{1}{2}\lambda - \psi\sin\tfrac{1}{2}\lambda)^2 + e^{4u}\sin^2\tfrac{1}{2}\lambda]^{-1}, \tag{2.21b}$$

$$\gamma' = \gamma. \tag{2.21c}$$

For the transform of ω see Geroch (1971). The asymptotic behaviour of these transformed potentials is given by

$$u' = -(m\cos\lambda)\rho^{-1} + O(\rho^{-2}), \tag{2.22a}$$

$$\psi' = (2m\sin\lambda)\rho^{-1} + O(\rho^{-2}), \tag{2.22b}$$

$$\omega' = \text{constant} - 2m\sin\lambda\cos\theta + O(\rho^{-1}). \tag{2.22c}$$

An additional trivial transformation which also enters our work is the replacement,

$$(e^{2u}, \omega, \psi) \to (K e^{2u}, K^{-1}\omega, K\psi), \tag{2.23}$$

where K is a constant.

3. Ordinary differential equations for H_4 and $K^{(\epsilon)}$

The metric coefficients and Ernst potentials may be expressed in a simple manner in terms of two functions, $H_4 = H_4(\eta)$ and $K^{(\epsilon)} = K^{(\epsilon)}(\nu, \eta)$, satisfying independent second-order ordinary DE in independent variables, η and ν, respectively. (Recall the definition (2.9) for η and ν.)

The DE for H_4 is

$$\eta^2(1+\eta)^2 H_4''^2 = 4H_4'(\eta H_4' - H_4)[-\delta^2 + H_4 - (1+\eta)H_4'], \tag{3.1}$$

where the prime denotes $\mathrm{d}/\mathrm{d}\eta$, subject to the boundary condition,

$$H_4 = \delta^2 p^{-2} + \mathrm{O}(\eta^{-1}) \qquad \text{as } \eta \to \infty. \tag{3.2}$$

δ and $p = (1-q^2)^{1/2}$ are arbitrary constants which will be identified with the corresponding parameters of TS (1973) in § 8. Attention will be mainly concentrated on the ranges, $0 < \delta < \infty$, $-1 < q < 1$, $0 < p \leq 1$. Regarding η as a complex variable, H_4 is an analytic function of η in the whole complex η plane, including $\eta = \infty$, except for a finite number of simple poles and, if δ is not an integer, two branch-point singularities at $\eta = 0$ and $\eta = -1$ requiring the plane to be cut from $\eta = 0$ to $\eta = -1$ along the real axis. The real simple poles, $\eta = \eta_0, \eta_1, \eta_2, \ldots, \eta_0 > \eta_1 > \eta_2 > \ldots$, will be seen, in § 4, to represent the infinite red-shift surfaces. An efficient method of solving the H_4 equation by infinite series is given in § 10.1. When δ is an integer, H_4 is a rational function of η and q^2.

Many later formulae are shortened considerably if, in terms of H_4, we define several other functions of η:

$$H_2 = (\operatorname{sgn} q)[-\delta^2 + H_4 - (1+\eta)H_4']^{1/2}, \tag{3.3}$$

$$\sigma_1 = \eta^{-1}(H_4 - \eta H_4')^{1/2}, \tag{3.4}$$

$$\sigma_2 = (\operatorname{sgn} q)(-H_4')^{1/2}, \tag{3.5}$$

$$\Lambda = \exp\left(\int_\eta^\infty (1+\tilde{\eta})^{-1/2}\sigma_1(\tilde{\eta})\,\mathrm{d}\tilde{\eta}\right), \tag{3.6}$$

$$\Gamma = \exp\left(-\int_\eta^\infty \tilde{\eta}^{-1}(1+\tilde{\eta})^{-1}(H_4(\tilde{\eta}) - \delta^2)\,\mathrm{d}\tilde{\eta}\right). \tag{3.7}$$

When δ is an integer, Γ is a polynomial in η^{-1} of degree δ^2; H_4, H_2, σ_1 and σ_2 are rational functions of η with $\eta^{\delta^2}\Gamma$ as denominator and Λ is a rational function of η and $(1+\eta)^{1/2}$. $\Gamma(\eta)$ is tabulated in the appendix for $\delta = 1, 2, 3, 4, 5$ and H_4, H_2, σ_1, σ_2 and Λ are tabulated for $\delta = 1, 2, 3$. For general δ and all $q \neq \pm 1$, Γ is analytic in the complex η plane (including $\eta = \infty$) cut from $\eta = -1$ to $\eta = 0$. The zeros of Γ are simple poles of H_4, H_2, σ_1 and σ_2 and are either simple poles or zeros of Λ. The signs of the square roots given in equations (3.3)–(3.6) are appropriate for the outer regions, $\eta > \eta_0$, and should be determined by analytic continuation for the inner regions.

The second ordinary DE, with ν as independent variable and η regarded as a constant parameter, is for either of two closely related functions, $K^{(\epsilon)} = K^{(\epsilon)}(\nu, \eta)$, where $\epsilon = +1$ or -1. It is:

$$K^{(\epsilon)}_{\nu\nu} + \left(-\frac{2\nu}{1-\nu^2} + \frac{\eta\nu}{1+\eta\nu^2} - \frac{\eta\sigma_1}{\eta\sigma_1\nu + \epsilon i\sigma_2}\right)K^{(\epsilon)}_\nu + \left(-\frac{\delta^2}{(1-\nu^2)^2} + \frac{H_4 + \epsilon i\eta H_2\nu}{(1-\nu^2)(1+\eta\nu^2)} - \frac{\epsilon i\eta\sigma_1 H_2}{(1-\nu^2)(\eta\sigma_1\nu + \epsilon i\sigma_2)}\right)K^{(\epsilon)} = 0. \quad (3.8)$$

This is a linear Fuchsian equation with five regular singular points and takes the form [2, 3, 0] in the Ince classification scheme (Ince 1927). Some properties of this equation and its solutions as well as alternative DE are discussed in § 10. When δ is an integer and (3.1) and (3.2) are satisfied, equation (3.8) admits polynomial solutions. The metric of space–time will depend on two linearly independent particular solutions of (3.8). These are the functions $K_1^{(\epsilon)}(\nu, \eta)$ and $K_2^{(\epsilon)}(\nu, \eta)$ defined by the boundary conditions:

$$K_1^{(\epsilon)} = 1, \qquad K_{1\nu}^{(\epsilon)} = \epsilon i((1+\eta)^{1/2}\sigma_2 - H_2) \qquad \text{at } \nu = 0 \quad (3.9a)$$

$$K_2^{(\epsilon)} = \epsilon i, \qquad K_{2\nu}^{(\epsilon)} = (1+\eta)^{1/2}\sigma_2 + H_2 \qquad \text{at } \nu = 0. \quad (3.9b)$$

They satisfy

$$\Delta \equiv K_1^{(\epsilon)}K_{2\nu}^{(\epsilon)} - K_2^{(\epsilon)}K_{1\nu}^{(\epsilon)} = 2(1+\eta)^{1/2}(1-\nu^2)^{-1}(1+\eta\nu^2)^{-1/2}(\sigma_2 - \epsilon i\eta\sigma_1\nu). \quad (3.10)$$

The close relation between $K^{(+1)}$ and $K^{(-1)}$ is shown by the following formulae:

$$K_1^{(\epsilon)}(\nu, \eta) = K_1^{(-\epsilon)}(-\nu, \eta), \quad (3.11a)$$

$$K_2^{(\epsilon)}(\nu, \eta) = -K_2^{(-\epsilon)}(-\nu, \eta), \quad (3.11b)$$

$$K^{(-\epsilon)} = -2\epsilon i\Delta^{-1}[K_\nu^{(\epsilon)} + \epsilon i H_2(1-\nu^2)^{-1}K^{(\epsilon)}]. \quad (3.12)$$

In all formulae below involving $K^{(\epsilon)}$, it is strongly advantageous to retain ϵ as a two-valued parameter rather than give ϵ one of its particular values, such as $\epsilon = +1$. In this way, all relations which are not invariant under $\epsilon \to -\epsilon$ actually split up into two independent relations. Throughout the rest of this paper, except where confusion may arise, the superscript '(ϵ)' will be dropped from $K^{(\epsilon)}$, $K_1^{(\epsilon)}$ and $K_2^{(\epsilon)}$.

The functions H_4, K_1, K_2 and those defined by (3.3)–(3.7) depend on three parameters, namely δ in (3.1), q in (3.2) and κ in (2.8) and (2.9). The space–time metrics constructed from these functions in § 4 will be the asymptotically flat metrics containing the discrete TS series but unphysical metrics with more parameters result from the same formulae if the definitions of H_4, K_1 and K_2 are generalised in the following manner. First, let H_4 be any solution of (3.1), i.e. not subject to (3.2). Then (3.3)–(3.7) still apply except that 'sgn q' may be replaced by ± 1 and $\int_\eta^\infty$ must be replaced by $\int_\eta = -\int^\eta$, i.e. minus any indefinite integral. Second, the boundary conditions (3.9) for K_1 and K_2 may be generalised by the replacement,

$$K_1 \to \alpha_1 K_1 + \alpha_2\Lambda^{-1}K_2, \qquad K_2 \to \alpha_3\Lambda K_1 + \alpha_4 K_2, \quad (3.13)$$

where $\alpha_1, \ldots, \alpha_4$ are constants and $\alpha_1\alpha_4 - \alpha_2\alpha_3 = 1$. These two generalisations each contribute one additional non-trivial parameter.

A third non-trivial parameter, h, may be introduced into the differential equations themselves. Let H_4 be any solution of

$$\eta^2(1+\eta)^2 H_4''^2 = 4(\eta H_4'^2 - H_4 H_4' - h^2)[-\delta^2 + H_4 - (1+\eta)H_4'] \quad (3.14)$$

which clearly reduces to (3.1) when $h=0$. Let H_2, σ_1, σ_2 and Γ be defined by (3.3), (3.4), (3.5) and (3.7) with the same comments as above applying, but Λ must be defined very differently (see § 5). Define also

$$H_1=\pm\tfrac{1}{2}(\eta^2\sigma_1^2\sigma_2^2-h^2)^{1/2} \tag{3.15}$$

with sign chosen so that (3.14) simplifies to

$$\eta(1+\eta)H_4''=4H_1H_2. \tag{3.16}$$

When $h=0$, $H_1=\tfrac{1}{2}\eta\sigma_1\sigma_2$. (Several other useful differential identities are given below in (5.7).) The equation for K must be modified to

$$K_{\nu\nu}+\left(-\frac{2\nu}{1-\nu^2}+\frac{\eta\nu}{1+\eta\nu^2}-\frac{\eta^2\sigma_1^2}{\eta^2\sigma_1^2\nu+2\epsilon\mathrm{i}H_1-h}\right)K_\nu+\left(-\frac{\delta^2}{(1-\nu^2)^2}+\frac{H_4+\epsilon\mathrm{i}\eta H_2\nu}{(1-\nu^2)(1+\eta\nu^2)}\right.$$
$$\left.-\frac{\epsilon\mathrm{i}\eta^2\sigma_1^2H_2}{(1-\nu^2)(\eta^2\sigma_1^2\nu+2\epsilon\mathrm{i}H_1-h)}+\frac{2h(1+\eta)\nu}{(1-\nu^2)^2(1+\eta\nu^2)}\right)K=0. \tag{3.17}$$

The boundary conditions for the particular solutions K_1 and K_2 are rather complicated when $h\neq 0$ and will be postponed till § 5. For the present, it is sufficient to note that

$$\Delta\equiv K_1K_{2\nu}-K_2K_{1\nu}$$
$$=2(1+\eta)^{1/2}(1-\nu^2)^{-1}(1+\eta\nu^2)^{-1/2}(\eta\sigma_1)^{-1}(-\epsilon\mathrm{i}\eta^2\sigma_1^2\nu+2H_1+\epsilon\mathrm{i}h), \tag{3.18}$$

and that, with this expression for Δ, equation (3.12) holds. It is interesting that, even when $h\neq 0$, there is a discrete series of δ values which lead to relatively simple elementary functional forms for the solutions of (3.14) and (3.17). These cases, which are very briefly mentioned at the end of § 11, reduce to the TS metrics, where δ is an integer, when $h=0$.

4. Construction of the metric and Ernst potentials from H_4 and K

The complex Ernst potentials, $\mathscr{E}$ and ξ, are not the most convenient for our purposes. Instead, from u and ψ, construct

$$F_1\equiv \mathrm{e}^{-2u},\qquad F_2\equiv-\psi\,\mathrm{e}^{-2u},\qquad F_3\equiv\mathrm{e}^{-2u}(\psi^2+\mathrm{e}^{4u}) \tag{4.1}$$

satisfying $F_1F_3-F_2^2=1$. Explicit formulae for the F are

$$F_1=\Lambda K_1^{(\epsilon)}K_1^{(-\epsilon)}=-2\epsilon\mathrm{i}\Lambda\Delta^{-1}[K_1K_{1\nu}+\epsilon\mathrm{i}H_2(1-\nu^2)^{-1}K_1^2]. \tag{4.2a}$$

$$F_2=\tfrac{1}{2}K_1^{(\epsilon)}K_2^{(-\epsilon)}+\tfrac{1}{2}K_2^{(\epsilon)}K_1^{(-\epsilon)}=-\epsilon\mathrm{i}\Delta^{-1}[K_1K_{2\nu}+K_2K_{1\nu}+2\epsilon\mathrm{i}H_2(1-\nu^2)^{-1}K_1K_2], \tag{4.2b}$$

$$F_3=\Lambda^{-1}K_2^{(\epsilon)}K_2^{(-\epsilon)}=-2\epsilon\mathrm{i}\Lambda^{-1}\Delta^{-1}[K_2K_{2\nu}+\epsilon\mathrm{i}H_2(1-\nu^2)^{-1}K_2^2]. \tag{4.2c}$$

The metric coefficients, ω and $\mathrm{e}^{2\gamma}$, may now be calculated by quadratures using (2.3) and (2.6*b*), respectively. Actually, both functions may be expressed in terms of functions already defined in § 3 without the need for quadratures. The F form three linearly independent solutions of a third-order linear DE which is investigated in § 7.

It is a far from straightforward matter to show by direct substitution into Ernst's equations that we actually have a vacuum gravitational field. The main difficulty arises in determining the behaviour of $K_1(\nu, \eta)$ and $K_2(\nu, \eta)$ with respect to η. The proof appears in § 6 after an efficient method of dealing with the η and ν behaviour on an equal basis is established. Similarly, it is very difficult to show directly that our solutions are asymptotically flat and are well behaved on the symmetry axis. This proof is given in § 8 with the aid of the F equation.

The formulae (4.2) apply also to the larger class of unphysical solutions with six non-trivial parameters, though, for $h \neq 0$, we have yet to define $\Lambda(\eta)$ and the appropriate boundary conditions for K_1 and K_2. Consider the effect of the replacement (3.13), which is applicable for all h, on F_1, F_2 and F_3. They transform as follows:

$$\begin{aligned} &F_1 \to \alpha_1^2 F_1 + 2\alpha_1\alpha_2 F_2 + \alpha_2^2 F_3, \\ &F_2 \to \alpha_1\alpha_3 F_1 + (\alpha_1\alpha_4 + \alpha_2\alpha_3) F_2 + \alpha_2\alpha_4 F_3, \\ &F_3 \to \alpha_3^2 F_1 + 2\alpha_3\alpha_4 F_2 + \alpha_4^2 F_3. \end{aligned} \tag{4.3}$$

This is easily seen to be a composition of a NUT transformation (2.21), a trivial transformation (2.23) and the replacement $\psi \to \psi + \text{constant}$. These three transformations form a group isomorphic to SL(2) and represented by the unimodular matrix,

$$\begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix}.$$

The case of a pure NUT transformation with NUT parameter λ is given by

$$\alpha_1 = \alpha_4 = \cos \tfrac{1}{2}\lambda, \qquad \alpha_2 = -\alpha_3 = \sin \tfrac{1}{2}\lambda. \tag{4.4}$$

The other two unphysical parameters, namely h and the one introduced by dropping the boundary condition (3.2), admit no such easy identification. They both destroy asymptotic flatness and create a curvature singularity along the symmetry axis.

The metric coefficient $e^{2\gamma}$ actually has a surprisingly simple formula. If $h = 0$, then

$$e^{2\gamma} = \left(1 + \frac{1}{\eta}\right)^{-\delta^2} \Gamma(\eta), \tag{4.5}$$

a function of η only, and if $h \neq 0$,

$$e^{2\gamma} = (\text{constant})\left(\frac{1+\nu}{1-\nu}\right)^{2h}\left(1 + \frac{1}{\eta}\right)^{-\delta^2} \Gamma(\eta). \tag{4.6}$$

These results are proved in § 6.

Let us now construct a Ricatti equation from the linear K equation. Then we shall be able to construct a compatible Ricatti equation in independent variable η with ν held constant. With the aid of a set of six identities, it will then be straightforward to show that the vacuum Einstein equations are satisfied as well as prove (4.5) or (4.6) for $e^{2\gamma}$ and (4.19), below, for ω. For the present, consider only $h = 0$ and $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (1, 0, 0, 1)$, but (3.2) need not necessarily be imposed.

From K_1 and K_2, construct three functions M_0, I_0 and J_0, independent of $\epsilon = \pm 1$, according to

$$K_1 = \left(\frac{\epsilon \mathrm{i}\sigma_2 + \eta\sigma_1\nu}{\epsilon \mathrm{i}\sigma_2 - \eta\sigma_1\nu}\right)^{1/4} I_0^{-1/2}[1+\epsilon \mathrm{i} M_0], \tag{4.7a}$$

$$K_2 = \left(\frac{\epsilon \mathrm{i}\sigma_2 + \eta\sigma_1\nu}{\epsilon \mathrm{i}\sigma_2 - \eta\sigma_1\nu}\right)^{1/4} I_0^{-1/2}[J_0+\epsilon \mathrm{i}(I_0 + M_0 J_0)]. \tag{4.7b}$$

Note that (4.7) splits into four equations because of the two values of ϵ. They are compatible on account of (3.10) and (3.12). The branches of the fourth roots are determined by making them take the value +1 when $\nu = 0$. On differentiating (4.7*a*, *b*), using (3.10) and (3.12), the M_0, I_0 and J_0 are seen to satisfy the following differential relations:

$$M_{0\nu} = A + BM_0^2, \tag{4.8}$$

$$I_{0\nu} = 2BM_0 I_0, \tag{4.9}$$

$$J_{0\nu} = -BI_0, \tag{4.10}$$

where

$$A = A(\nu, \eta) = \frac{R^{1/2}}{1-\nu^2} + \left(\frac{1+\eta}{1+\eta\nu^2}\frac{H_1}{R} - \frac{H_2}{1-\nu^2}\right), \tag{4.11}$$

$$B = B(\nu, \eta) = -\frac{R^{1/2}}{1-\nu^2} + \left(\frac{1+\eta}{1+\eta\nu^2}\frac{H_1}{R} - \frac{H_2}{1-\nu^2}\right), \tag{4.12}$$

where $H_1 = \frac{1}{2}\eta\sigma_1\sigma_2$ and

$$R = R(\nu, \eta) = (1+\eta)(1+\eta\nu^2)^{-1}(\sigma_2^2 + \eta^2\sigma_1^2\nu^2). \tag{4.13}$$

The transformations (3.13) or (4.3) correspond to

$$M_0 \to M = M_0 + I_0(J_0 + \alpha_1\alpha_2^{-1}\Lambda)^{-1}, \tag{4.14a}$$

$$I_0 \to I = \Lambda^2 I_0(\alpha_2 J_0 + \alpha_1\Lambda)^{-2}, \tag{4.14b}$$

$$J_0 \to J = (\alpha_4\Lambda J_0 + \alpha_3\Lambda^2)(\alpha_2 J_0 + \alpha_1\Lambda)^{-1}. \tag{4.14c}$$

The functions M, I and J satisfy the same differential relations as M_0, I_0 and J_0. Note particularly that M satisfies the Ricatti equation,

$$M_\nu = A + BM^2, \tag{4.15}$$

and that (4.14*a*) shows how the general solution of (4.15) is constructed from a particular solution $M = M_0$ (interpreting $\alpha_1\alpha_2^{-1}\Lambda$ as the arbitrary constant of integration). It is clear from (3.9) and (4.7) that M_0, I_0 and J_0 are distinguished by the following simple boundary conditions at $\nu = 0$:

$$M_0(0, \eta) \equiv 0, \qquad I_0(0, \eta) \equiv 1, \qquad J_0(0, \eta) \equiv 0. \tag{4.16}$$

In terms of M_0, I_0 and J_0, the explicit formulae for the metric coefficients and Ernst potentials take the form:

$$\mathrm{e}^{2u} = \Lambda^{-1} I_0 (1+M_0^2)^{-1}, \tag{4.17}$$

$$\psi = -\Lambda^{-1} I_0 M_0 (1+M_0^2)^{-1} - \Lambda^{-1} J_0, \tag{4.18}$$

$$\omega = 2\kappa\delta q p^{-1} + 2\kappa\Lambda I_0^{-1}[R^{1/2}(1-M_0^2) - H_2(1+M_0^2)]. \tag{4.19}$$

Equations (4.17) and (4.18) follow directly from (4.1), (4.2) and (4.7). This explicit formula for ω will be shown in the next section to satisfy equations (2.3). The additive constant $2\kappa\delta qp^{-1}$ in (4.19) is chosen so that $\omega\to 0$ as $\eta\to\infty$ when (3.2) applies, $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)=(1, 0, 0, 1)$ and $h=0$. In all other cases, an arbitrary additive constant may be substituted.

In practice, ω should not be computed directly from equation (4.19). A much better formula,

$$\omega=\kappa[2\delta qp^{-1}-2H_2\,\mathrm{e}^{-2u}-(1-\nu^2)\psi_\nu\,\mathrm{e}^{-4u}], \tag{4.20}$$

follows immediately from (4.17)–(4.19) and (4.8)–(4.10). In fact, the functions M_0, I_0 and J_0, though of considerable theoretical utility, are very poorly suited to investigation of the physical properties of space–time, especially in the asymptotically flat outer regions. The difficulties arise from the fact that $R^{1/2}$ is directionally singular at infinity. A single simple example will suffice to show this singular behaviour. A direct substitution from (2.16) and the appendix gives for the Kerr metric, $\delta=1$, in spheroidal coordinates,

$$R^{1/2}=(p^2x^2+q^2y^2-1)^{-1}[p^2y^2(x^2-1)^2+q^2x^2(1-y^2)^2]^{1/2}, \tag{4.21}$$

$$\begin{aligned}M_0=y^{-1}(px^2+2x+p)^{-1}(p^2x^2+q^2y^2-1)^{-1}\{&-q^3xy^4+qy^2(-p^2x^3+p^2x+2x+2p)\\&-qx(px+1)^2+[(px+1)^2+q^2y^2][p^2y^2(x^2-1)^2+q^2x^2(1-y^2)^2]^{1/2}\}.\end{aligned} \tag{4.22}$$

Clearly

$$\lim_{y\to 0}\lim_{x\to\infty}\neq\lim_{x\to\infty}\lim_{y\to 0}$$

for many expressions involving $R^{1/2}$, M_0, I_0 or J_0 and we shall say that these functions suffer from the 'R singularity'. The functions K_1, K_2, F_1, F_2 and F_3 are all free of this 'R singularity', though it appears in a more or less innocuous fashion in the DE (3.8). (See §§ 7 and 10.)

The formulae (4.17)–(4.19) are, however, useful for the equatorial plane $\nu=0$. On $\nu=0$, $\psi=0$ and

$$\mathrm{e}^{2u}=\Lambda^{-1}, \qquad \omega=2\kappa\delta qp^{-1}+2\kappa\Lambda[(1+\eta)^{1/2}\sigma_2-H_2]. \tag{4.23}$$

A knowledge of the metric coefficients on the equatorial plane only is sufficient for many investigations. The functions in (4.23) may be efficiently computed by the methods of § 10 even in parts of the highly curved inner regions of space–time. Note that the real zeros of $\Gamma(\eta)$, say $\eta=\eta_0, \eta_1, \eta_2, \ldots$ (arranged so that $x_0>x_1>x_2>\ldots$, running into negative values, where $\eta_i=x_i^2-1$ for $y=0$, the sequence terminating at $-x_0$) are simple poles of H_4, H_2, σ_1 and σ_2 and are alternately simple poles or simple zeros of Λ. In particular, $\eta_0, \eta_2, \eta_4, \ldots$ are simple poles and $\eta_1, \eta_3, \eta_5, \ldots$ are simple zeros of Λ. The simple zeros of Λ represent ring shaped curvature singularities on the equatorial plane. These ring singularities are well known in the cases of the Kerr and TS metrics. TS (1973) noted, without proof, that the space–times with δ an integer have exactly 2δ infinite red-shift surfaces and that the ring singularities on the equatorial plane reside on every second such surface.

Actually, the full $(2+1)$-surfaces, $\eta=\eta_0, \eta=\eta_1, \ldots$, each having topology $S_2\times R_1$, are precisely the infinite red-shift surfaces of space–time. On these surfaces, the metric and Ernst potentials assume familiar transcendental functional forms (of the coordinates, but not the parameters). When the limit $\eta \to \eta_0$ is taken in the DE (3.8), all the coefficients remain finite. In terms of the constants,

$$S_0 = \Gamma'(\eta_0), \qquad T_0 = \Gamma''(\eta_0), \qquad \Lambda_0 = \lim_{\eta\to\eta_0} (\eta - \eta_0)\Lambda(\eta), \tag{4.24}$$

and a new dependent variable,

$$\bar{K}(\nu) = (1 - \epsilon i \eta_0^{1/2}\nu)^{-1/2} K^{(\epsilon)}(\nu, \eta_0) = (1 + \epsilon i \eta_0^{1/2}\nu)^{-1/2} K^{(-\epsilon)}(\nu, \eta_0) \tag{4.25}$$

(independent of ϵ—consistent with (3.12)), the limiting equation reads:

$$\bar{K}_{\nu\nu} + \left(-\frac{2\nu}{1-\nu^2} + \frac{\eta_0\nu}{1+\eta_0\nu^2}\right)\bar{K}_\nu + \left(\frac{\eta_0}{1+\eta_0\nu^2} - \frac{\delta^2}{(1-\nu^2)^2} + \frac{\frac{1}{2}\eta_0(1+\eta_0)T_0S_0^{-1} - \delta^2\eta_0}{(1-\nu^2)(1+\eta_0\nu^2)}\right)\bar{K} = 0. \tag{4.26}$$

This is precisely a Lamé equation for $(1-\nu^2)^{1/4}\bar{K}$ in the form [2, 2, 0] in the Ince (1927) classification scheme. The substitution $\mu = \nu^2$ puts it into the more familiar [3, 1, 0] form. The particular solutions,

$$\bar{K}_1(\nu) = (1 - \epsilon i \eta_0^{1/2}\nu)^{-1/2} K_1(\nu, \eta_0),$$

$$\bar{K}_2(\nu) = (1 - \epsilon i \eta_0^{1/2}\nu)^{-1/2}\left(\lim_{\eta\to\eta_0} (\eta - \eta_0) K_2(\nu, \eta)\right),$$

satisfy the boundary conditions,

$$\bar{K}_1 = 1, \qquad \bar{K}_{1\nu} = 0, \qquad \bar{K}_2 = 0, \qquad \bar{K}_{2\nu} = 2\eta_0^{1/2}(1+\eta_0),$$

at $\nu = 0$, implying that $\bar{K}_1$ is even, $\bar{K}_2$ odd in ν. The explicit formulae for e^{2u}, ψ and ω on the surface $\eta = \eta_0$ are given by

$$\lim_{\eta\to\eta_0} (\eta - \eta_0)^{-1} e^{2u} = \Lambda_0^{-1}(1+\eta_0\nu^2)^{-1/2}(\bar{K}_1(\nu))^{-2}, \tag{4.27a}$$

$$\psi = -\Lambda_0^{-1}\bar{K}_2(\nu)(\bar{K}_1(\nu))^{-1}, \tag{4.27b}$$

$$\omega\, e^{2u} = -\kappa\eta_0^{1/2}(1-\nu^2)(1+\eta_0\nu^2)^{-1}. \tag{4.27c}$$

Equally simple formulae apply for the normal derivatives of these functions and for the metric coefficient, $e^{-2u}(\omega^2 e^{4u} - r^2)$.

The corresponding formulae for the infinite red-shift surfaces, $\eta = \eta_2$, $\eta = \eta_4, \ldots$, are identical to the above (if $\eta_{2i} > 0$) if we replace S_0 by $S_2 = \Gamma'(\eta_2)$, and so on. The cases, $\eta = \eta_1$, $\eta = \eta_3, \ldots$, which carry the ring singularities, need to be treated slightly differently. Similarly, minor adjustments are needed for those cases where $\eta < 0$ and/or $x < 0$ which are of interest only in the case when δ is an integer. Consider $\eta = \eta_1$, supposing $\eta_1 > 0$ and $x > 0$ on $\eta = \eta_1$ (true at least for $\delta \geq 2$). Then equations (4.25) and (4.26) remain unchanged except for the obvious replacement of η_0, S_0, T_0 by η_1, $S_1 = \Gamma'(\eta_1)$, $T_1 = \Gamma''(\eta_1)$. The particular solutions $\bar{K}_1(\nu)$ and $\bar{K}_2(\nu)$ are defined somewhat differently by

$$\bar{K}_1(\nu) = \epsilon i(1 - \epsilon i \eta_1^{1/2}\nu)^{-1/2}\left(\lim_{\eta\to\eta_1} (\eta - \eta_1) K_1(\nu, \eta)\right),$$

$$\bar{K}_2(\nu) = -\epsilon i(1 - \epsilon i \eta_1^{1/2}\nu)^{-1/2} K_2(\nu, \eta_1),$$

and satisfy boundary conditions,

$$\bar{K}_1=0, \qquad \bar{K}_{1\nu}=2\eta_1^{1/2}(1+\eta_1), \qquad \bar{K}_2=1, \qquad \bar{K}_{2\nu}=0,$$

at $\nu=0$. Then, with

$$\Lambda_1=\lim_{\eta\to\eta_1}(\eta-\eta_1)^{-1}\Lambda(\eta),$$

the formulae for e^{2u}, ψ and ω take the form,

$$\lim_{\eta\to\eta_1}(\eta-\eta_1)^{-1}\,e^{2u}=\Lambda_1^{-1}(1+\eta_1\nu^2)^{-1/2}(\bar{K}_1(\nu))^{-2}, \tag{4.28a}$$

$$\psi=\Lambda_1^{-1}\bar{K}_2(\nu)(\bar{K}_1(\nu))^{-1}, \tag{4.28b}$$

$$\omega\,e^{2u}=-\kappa\eta_1^{1/2}(1-\nu^2)(1+\eta_1\nu^2)^{-1}. \tag{4.28c}$$

Note that e^{2u} vanishes on all the surfaces, $\eta=\eta_0$, $\eta=\eta_1,\ldots$, except on the equatorial plane ($\nu=0$) in the cases, $\eta=\eta_1$, $\eta=\eta_3,\ldots$, where it is singular. The vanishing of e^{2u} is the criterion for an infinite red-shift surface which is the boundary of an ergosphere ($e^{2u}\leqslant 0$).

Let us conclude this section with a discussion of the various possible singularities of our (asymptotically flat) space–times. If δ is not an integer, then H_4, H_2, σ_1, σ_2 and Λ have complicated branch point singularities at $\eta=0$ and $\eta=-1$. Consequently, the $(2+1)$-surface, $x=1$, is a natural boundary for the exterior vacuum metric. Outside $x=1$, there are no other singularities apart from the ring singularities already discussed (if any with $x>1$). The most important use of these solutions in astrophysics would be as vacuum exteriors of finite rotating bodies, e.g. neutron stars, whose mass and angular momentum multipole moments depend on precisely three parameters.

However, if δ is an integer (and $q\neq 0$), one may meaningfully discuss the highly curved inner regions beyond $x=1$. The functions, H_4, H_2, σ_1, σ_2 and Λ are now analytic at $\eta=0$ and $\eta=-1$ (except that Λ is an analytic function of $(1+\eta)^{1/2}$ at $\eta=-1$). In the case $\delta=1$ (Kerr metric), the surfaces $x=1$ and $x=-1$ are the non-singular event horizons. For $\delta\geqslant 2$, TS (1972, 1973) also interpreted these surfaces as event horizons, but this is not strictly correct (see Gibbons and Russell-Clark 1973, Glass 1973). In these cases, the poles, $x^2=1$, $y^2=1$, appear as directional singularities. Actually, for $\delta=2$, Ernst (1976) and Economou (1976) have shown that these 'points' are non-singular surfaces as suggested by the calculation of the Weyl tensor (Economou and Ernst 1976). Without doubt, the main conclusions of these two authors will apply for $\delta\geqslant 3$, except that the interesting wormhole topology will increase in complexity. Of course, these solutions cannot be considered as 'black holes' because of the naked ring singularities. Nevertheless, they deserve some consideration as possible end-points of gravitational collapse since the non-formation of naked singularities is still conjecture.

5. Ricatti equations for M_0; η dependence

In the last section, we saw that (for $h=0$) the function $M_0(\nu,\eta)$ was a particular solution of the Ricatti equation,

$$M_\nu=A+BM^2, \tag{5.1}$$

satisfying $M_0(0,\eta)\equiv 0$. The general solution is given by (4.14a) if $\alpha_1\alpha_2^{-1}\Lambda$ is interpreted as the arbitrary 'constant' of integration. The coefficients A and B are given by

(4.11) and (4.12). Now, we shall prove that M, as defined by (4.14a), satisfies the Ricatti equation in independent variable η,

$$M_\eta = C + DM + EM^2, \tag{5.2}$$

where

$$D = D(\nu, \eta) = 2\eta^{-1} H_1 R^{-1/2}, \tag{5.3}$$

$$\begin{Bmatrix} C \\ E \end{Bmatrix} = \begin{Bmatrix} C(\nu, \eta) \\ E(\nu, \eta) \end{Bmatrix} = \frac{H_4 \nu}{2\eta(1+\eta\nu^2)R}(\mp R^{1/2} + H_2), \tag{5.4), (5.5}$$

minus sign for C, plus for E. Considering the identity, $M_{\nu\eta} \equiv M_{\eta\nu}$, we see that compatibility of (5.1) and (5.2) is guaranteed by

$$\begin{aligned} A_\eta - C_\nu - DA &= 0, \\ D_\nu - 2BC + 2EA &= 0, \\ B_\eta - E_\nu + DB &= 0, \end{aligned} \tag{5.6}$$

which may be proved by direct substitution. (Note: the following identities, which follow directly from (3.3)–(3.6), (3.15) and (3.16), are most useful in these calculations:

$$H_4' = -\sigma_2^2, \qquad H_2' = -2\eta^{-1} H_1, \tag{5.7a, b}$$

$$H_1' = -\tfrac{1}{2}(1+\eta)^{-1} H_2(\sigma_2^2 + \eta\sigma_1^2), \qquad \Lambda' = -(1+\eta)^{-1/2}\sigma_1\Lambda, \tag{5.7c, d}$$

$$\sigma_1' = -\frac{\sigma_1}{\eta} - \frac{2H_1H_2}{\eta^2(1+\eta)\sigma_1}, \qquad \sigma_2' = -\frac{2H_1H_2}{\eta(1+\eta)\sigma_2}. \tag{5.7e, f}$$

Except for (5.7d) for Λ', these equations are written in a form valid also for $h \neq 0$.) Now the identities (5.6) are not quite sufficient to guarantee that M satisfies (5.2) when the functional form (3.6) of Λ is taken into account. If (5.6) are solved for unknowns C, D and E, then the solutions are determined up to three arbitrary functions of η. However, two of these are completely arbitrary—only the third is determined by the functional form of Λ. So, to complete the proof of (5.2), merely set $\nu = 0$ in (5.2), using (4.14a), (4.16), (5.3)–(5.5), to obtain

$$\Lambda' = -D(0, \eta)\Lambda. \tag{5.8}$$

This is in agreement with the definition (3.6) or (5.7d).

It is now a simple matter to verify the important identities,

$$M_{0\eta} = C + DM_0 + EM_0^2, \tag{5.9}$$

$$(\Lambda^{-1} I_0)_\eta = (D + 2EM_0)\Lambda^{-1} I_0, \tag{5.10}$$

$$(\Lambda^{-1} J_0)_\eta = -E\Lambda^{-1} I_0, \tag{5.11}$$

which are satisfied also by M, I and J. The theoretical importance of the functions M_0, I_0 and J_0 rests on the set of six identities (4.8)–(4.10) and (5.9)–(5.11). The proof that Einstein's vacuum equations are satisfied is now straightforward, though rather lengthy (see § 6). Consider, for example, the formulae (4.17), (4.18) and (4.19) for e^{2u}, ψ and ω. In (ν, η) coordinates, the relations (2.3) read:

$$\psi_\nu = -\frac{2(1+\eta)(1+\eta\nu^2)}{\kappa(1-\nu^2)^2} e^{4u}\omega_\eta, \qquad \psi_\eta = \frac{1+\eta\nu^2}{2\kappa\eta(1+\eta)} e^{4u}\omega_\nu. \tag{5.12}$$

These may now be proved by direct substitution using the relations (4.8)–(4.10) and (5.9)–(5.11).

Let us now derive the appropriate boundary conditions for K satisfying the DE (3.17) for the cases $h \neq 0$. First, consider the two compatible Ricatti equations,

$$M_\nu = A + BM^2, \qquad M_\eta = C + DM + EM^2, \tag{5.13}$$

where A, B and D are defined, as before, by (4.11), (4.12) and (5.3) with the definition of R modified to

$$R = (1+\eta)(1+\eta\nu^2)^{-1}(\sigma_2^2 + \eta^2\sigma_1^2\nu^2 - 2h\nu),$$

and the definitions of C and E are modified to

$$\begin{Bmatrix} C \\ E \end{Bmatrix} = \frac{H_4\nu - h(1-\eta\nu^2)}{2\eta(1+\eta\nu^2)R}(\mp R^{1/2} + H_2).$$

When $h \neq 0$, we cannot define a particular solution, $M = M_0$, satisfying $M_0(0, \eta) \equiv 0$. Consequently, we must choose M_0 to be the particular solution of (5.13) satisfying

$$M_0(0, \eta) \equiv \mu_0(\eta) \tag{5.14}$$

where μ_0 is any particular solution of the Ricatti equation,

$$\mu' = C(0, \eta) + D(0, \eta)\mu + E(0, \eta)\mu^2. \tag{5.15}$$

Now, the general solution of (5.15) is

$$\mu = \mu_0 + \mu_1/(\mu_2 + c), \tag{5.16}$$

where c is the arbitrary constant of integration and

$$\mu_1 = \exp\left(\int_a^\eta (D(0, \tilde\eta) + 2E(0, \tilde\eta)\mu_0(\tilde\eta))\,d\tilde\eta\right), \tag{5.17a}$$

$$\mu_2 = -\int_a^\eta E(0, \tilde\eta)\mu_1(\tilde\eta)\,d\tilde\eta, \tag{5.17b}$$

where a is a fixed constant. The general solution of (5.13) now reads,

$$M = M_0 + I_0(J_0 + \alpha_1\alpha_2^{-1}\Lambda)^{-1}$$

where $\alpha_1\alpha_2^{-1}$ is the constant of integration and

$$I_0 = \exp\left(\int_0^\nu 2B(\tilde\nu, \eta)M_0(\tilde\nu, \eta)\,d\tilde\nu\right), \tag{5.18a}$$

$$J_0 = -\int_0^\nu B(\tilde\nu, \eta)I_0(\tilde\nu, \eta)\,d\tilde\nu + \mu_2\mu_1^{-1}, \tag{5.18b}$$

$$\Lambda = \mu_1^{-1}. \tag{5.19}$$

The M_0, I_0, J_0 and Λ defined in this way satisfy the identities, (4.8)–(4.10) and (5.9)–(5.11). So also do M, I and J as defined by equations (4.14).

Now construct e^{2u}, ψ and ω by equations (4.17), (4.18) and (4.19), respectively (noting that the additive constant, $2\kappa\delta qp^{-1}$, in (4.19) may be replaced by an arbitrary constant). Clearly, the relations, (2.3), or, equivalently, (5.12), are satisfied. Next, construct

$$K_1=\left(\frac{2H_1+\epsilon ih-\epsilon i\eta^2\sigma_1^2\nu}{2H_1-\epsilon ih+\epsilon i\eta^2\sigma_1^2\nu}\right)^{1/4} I_0^{-1/2}(1+\epsilon \mathrm{i}M_0), \tag{5.20a}$$

$$K_2=\left(\frac{2H_1+\epsilon ih-\epsilon i\eta^2\sigma_1^2\nu}{2H_1-\epsilon ih+\epsilon i\eta^2\sigma_1^2\nu}\right)^{1/4} I_0^{-1/2}[J_0+\epsilon \mathrm{i}(I_0+M_0J_0)]. \tag{5.20b}$$

It may now be immediately verified that K_1 and K_2 satisfy the differential equation (3.17) and the relations (3.18) and (3.12), and that e^{2u} and ψ are given by the formulae (4.2). The boundary conditions satisfied by K_1 and K_2 at $\nu=0$ are:

$$K_1=\theta(1+\epsilon \mathrm{i}\mu_0), \tag{5.21a}$$

$$K_{1\nu}=\theta[\mu_0((1+\eta)^{1/2}\sigma_2+H_2)+\epsilon \mathrm{i}((1+\eta)^{1/2}\sigma_2-H_2)], \tag{5.21b}$$

$$K_2=\theta\mu_1^{-1}[\mu_2+\epsilon \mathrm{i}(\mu_1+\mu_0\mu_2)], \tag{5.21c}$$

$$K_{2\nu}=\theta\mu_1^{-1}\{(\mu_1+\mu_0\mu_2)[(1+\eta)^{1/2}\sigma_2+H_2]+\epsilon \mathrm{i}\mu_2[(1+\eta)^{1/2}\sigma_2-H_2]\}, \tag{5.21d}$$

where

$$\theta=(\eta\sigma_1\sigma_2)^{-1/2}(2H_1+\epsilon ih)^{1/2}.$$

Note that when $h\neq 0$, the explicit formulae for the metric coefficients depend on three, rather than two, ordinary DE. The third is the Ricatti equation (5.15), which we shall not discuss further in this paper. When δ takes the discrete series of values (see (11.22), below) for which the H_4 and K equations admit elementary functional solutions, the Ricatti equation (5.15) also admits such solutions.

6. The '*H*' equations; proof that Einstein's and Ernst's equations are satisfied and comparison with Tomimatsu–Sato 'rule (*a*)'

In (ν, η) coordinates, Einstein's equations take the form:

$$E_1\equiv\tfrac{1}{4}(1-\nu^2)^2u_{\nu\nu}+\eta(1+\eta)^2u_{\eta\eta}-\frac{\frac{1}{2}\nu(1-\nu^2)(2+\eta+\eta\nu^2)}{1+\eta\nu^2}u_\nu+\frac{(1+\eta)(1+2\eta+\eta^2\nu^2)}{1+\eta\nu^2}u_\eta$$
$$+\frac{(1+\eta\nu^2)^2}{2\kappa^2\eta(1-\nu^2)^2}e^{4u}[\tfrac{1}{4}(1-\nu^2)^2\omega_\nu^2+\eta(1+\eta)^2\omega_\eta^2]=0, \tag{6.1}$$

$$E_2\equiv\tfrac{1}{4}(1-\nu^2)^2\omega_{\nu\nu}+\eta(1+\eta)^2\omega_{\eta\eta}+\frac{\frac{1}{2}\eta\nu(1-\nu^2)^2}{1+\eta\nu^2}\omega_\nu+\frac{\eta(1+\eta)[1+(1+2\eta)\nu^2]}{1+\eta\nu^2}\omega_\eta$$
$$+(1-\nu^2)^2u_\nu\omega_\nu+4\eta(1+\eta)^2u_\eta\omega_\eta=0. \tag{6.2}$$

We have already mentioned that these can be proved by direct substitution from (4.17) and (4.19), using the six identities (4.8)–(4.10) and (5.9)–(5.11). Actually, the relations (5.12), whose proof requires much less labour, imply $E_2=0$ since

$$E_2\equiv\tfrac{1}{2}\kappa\eta(1+\eta)(1+\eta\nu^2)^{-1}(1-\nu^2)^2\,e^{-4u}\left(\frac{\partial}{\partial\nu}(\psi_\eta)-\frac{\partial}{\partial\eta}(\psi_\nu)\right)=0. \tag{6.3}$$

Similarly, a direct substitution from those same identities will prove the following interesting formulae:

$$H_1(\eta)=\frac{1+\eta\nu^2}{\kappa(1+\eta)(1-\nu^2)}\,e^{2u}[\tfrac{1}{4}(1-\nu^2)^2u_\nu\omega_\nu+\eta(1+\eta)^2u_\eta\omega_\eta], \tag{6.4}$$

$$H_2(\eta)=\frac{(1+\eta)(1+\eta\nu^2)}{\kappa(1-\nu^2)}\,e^{2u}\omega_\eta-\frac{1}{2\kappa}\,e^{2u}(\omega-2\kappa\delta qp^{-1}), \tag{6.5}$$

$$\begin{aligned}H_3(\eta)&\equiv\eta^2\sigma_1^2+\eta\sigma_2^2\\&=\frac{4\eta}{1+\eta}[\tfrac{1}{4}(1-\nu^2)^2u_\nu^2+\eta(1+\eta)^2u_\eta^2]\\&\quad+\frac{(1+\eta\nu^2)^2}{\kappa^2(1+\eta)(1-\nu^2)^2}\,e^{4u}[\tfrac{1}{4}(1-\nu^2)^2\omega_\nu^2+\eta(1+\eta)^2\omega_\eta^2],\end{aligned} \tag{6.6}$$

$$\begin{aligned}H_4(\eta)&\equiv\eta^2\sigma_1^2-\eta\sigma_2^2\\&=-\frac{4\eta(1-\eta\nu^2)}{(1+\eta)(1+\eta\nu^2)}[\tfrac{1}{4}(1-\nu^2)^2u_\nu^2-\eta(1+\eta)^2u_\eta^2]\\&\quad+\frac{1-\eta^2\nu^4}{\kappa^2(1+\eta)(1-\nu^2)^2}\,e^{4u}[\tfrac{1}{4}(1-\nu^2)^2\omega_\nu^2-\eta(1+\eta)^2\omega_\eta^2]\\&\quad-\frac{8\eta^2\nu(1-\nu^2)}{1+\eta\nu^2}u_\nu u_\eta+\frac{2\eta\nu(1+\eta\nu^2)}{\kappa^2(1-\nu^2)}\,e^{4u}\omega_\nu\omega_\eta,\end{aligned} \tag{6.7}$$

$$\begin{aligned}h&=-\frac{4\eta\nu}{(1+\eta)(1+\eta\nu^2)}[\tfrac{1}{4}(1-\nu^2)^2u_\nu^2-\eta(1+\eta)^2u_\eta^2]\\&\quad+\frac{\nu(1+\eta\nu^2)}{\kappa^2(1+\eta)(1-\nu^2)^2}\,e^{4u}[\tfrac{1}{4}(1-\nu^2)^2\omega_\nu^2-\eta(1+\eta)^2\omega_\eta^2]\\&\quad+\frac{2\eta(1-\nu^2)(1-\eta\nu^2)}{1+\eta\nu^2}u_\nu u_\eta-\frac{1-\eta^2\nu^4}{2\kappa^2(1-\nu^2)}\,e^{4u}\omega_\nu\omega_\eta.\end{aligned} \tag{6.8}$$

These formulae are valid for all h, all $(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$ with $\alpha_1\alpha_4-\alpha_2\alpha_3=1$ and for any solution of the H_4 equation (3.14). Note that the left-hand sides of (6.4)–(6.8) are independent of $(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$. In fact, it is easy to show that if any stationary axisymmetric vacuum solution of Einstein's equations is substituted into the right-hand sides of (6.4), (6.6), (6.7) and (6.8), then these expressions are invariant under the transformations (4.3) which contain the NUT transformation. This general theorem is not true for the right-hand side of (6.5).

Now the metric coefficient $e^{2\gamma}$ is obtained by quadratures from (2.6*a*). Converting to (ν,η) coordinates and comparing with (6.7) and (6.8), we have the remarkable results,

$$\gamma_\nu=2h(1-\nu^2)^{-1},\qquad \gamma_\eta=\tfrac{1}{2}\eta^{-1}(1+\eta)^{-1}H_4. \tag{6.9}$$

Compatibility is obvious and the expression (4.6) for $e^{2\gamma}$ follows immediately. Note that when $h=0$,

$$e^{2\gamma}=(1+1/\eta)^{-\delta^2}\Gamma(\eta), \tag{6.10}$$

a function of η only. When $\delta = 1, 2, 3, 4$, this result may be directly compared with the results of Tomimatsu and Sato (1973). They find

$$e^{2\gamma} = (\text{constant})(x^2 - y^2)^{-\delta^2} A(x, y)$$

where $A(x, y)$ is a polynomial in x and y of degree $2\delta^2$. Inspection of their tabulated forms of $A(x, y)$ for $\delta = 1, 2, 3$ (and when $\delta = 4$ it is easy to construct $A(x, y)$ from their tabulated form for ξ) shows that it is a homogeneous polynomial in the variables, $x^2 - 1$ and $1 - y^2$. Thus $e^{2\gamma}$ is a function of $(x^2-1)/(1-y^2) = \eta$ only. In the appendix, we tabulate $\Gamma(\eta)$ for $\delta = 1, 2, 3, 4, 5$ and, in the first four cases, our expressions are in precise agreement with those of TS.

The complete proof of Einstein's equations may now be deduced from (6.3) and the compatibility of (6.9). From (6.9), (6.7) and (6.8),

$$\begin{aligned}\frac{\partial}{\partial\eta}(\gamma_\nu) - \frac{\partial}{\partial\nu}(\gamma_\eta) &\\ \equiv \frac{4}{(1+\eta)(1+\eta\nu^2)}\left(\frac{1-\eta\nu^2}{1+\eta}u_\nu + \frac{4\eta\nu}{1-\nu^2}u_\eta\right)E_1 &\\ - \frac{1+\eta\nu^2}{\kappa^2\eta(1+\eta)(1-\nu^2)^2}\left(\frac{1-\eta\nu^2}{1+\eta}\omega_\nu + \frac{4\eta\nu}{1-\nu^2}\omega_\eta\right) e^{4u}E_2 = 0. & \qquad (6.11)\end{aligned}$$

Similarly, from (6.6), (6.7), (6.8) and (3.5)

$$\begin{aligned}\frac{1-\nu^2}{1+\eta}\frac{\partial}{\partial\nu}(h) + \frac{\partial}{\partial\eta}(H_4) + \frac{1}{2\eta}H_3 - \frac{1}{2\eta}H_4 &\\ \equiv \frac{8\eta(1-\nu^2)}{(1+\eta)(1+\eta\nu^2)}\left(-\frac{\nu}{1+\eta}u_\nu + \frac{1-\eta\nu^2}{1-\nu^2}u_\eta\right)E_1 &\\ - \frac{2(1+\eta\nu^2)}{\kappa^2(1+\eta)(1-\nu^2)}\left(-\frac{\nu}{1+\eta}\omega_\nu + \frac{1-\eta\nu^2}{1-\nu^2}\omega_\eta\right) e^{4u}E_2 = 0. & \qquad (6.12)\end{aligned}$$

This last result is equivalent to the Einstein equation,

$$\gamma_{rr} + \gamma_{zz} + (1/r)\gamma_r = -2u_z^2 - (1/2r^2)\, e^{4u}\omega_r^2.$$

Of course, with (6.1) and (5.12) now known to be satisfied, Ernst's equations follow immediately.

Equation (6.8) may be compared with the TS 'rules for computation'. Converting (6.8) to (x, y) coordinates and using (2.3) and (2.5*a*, *b*), we get

$$\begin{aligned} h &= (x^2-1)(1-y^2)(u_x u_y + \tfrac{1}{4} e^{-4u}\psi_x\psi_y) \\ &= \tfrac{1}{8}(x^2-1)(1-y^2)\, e^{-4u}(\mathscr{E}_x\mathscr{E}_y^* + \mathscr{E}_x^*\mathscr{E}_y) \\ &= \tfrac{1}{32}(x^2-1)(1-y^2)\, e^{-4u}(1-\mathscr{E})^2(1-\mathscr{E}^*)^2(\xi_x\xi_y^* + \xi_x^*\xi_y), \end{aligned} \qquad (6.13)$$

where the asterisk denotes complex conjugate. Now TS write ξ as the ratio of two complex polynomials, α and β: $\xi = \alpha/\beta$. Their 'rule (*a*)' states:

$$\text{Im}(\beta\alpha_x - \alpha\beta_x) = 0, \qquad \text{Re}(\beta\alpha_y - \alpha\beta_y) = 0.$$

Since, for general δ, there is no obviously preferred numerator and denominator, we combine these two relations to form a single relation for ξ:

$$\xi_x\xi_y^* + \xi_x^*\xi_y = 2\beta^{-2}\beta^{*-2}[\mathrm{Re}(\beta\alpha_x - \alpha\beta_x)\,\mathrm{Re}(\beta\alpha_y - \alpha\beta_y) + \mathrm{Im}(\beta\alpha_x - \alpha\beta_x)\,\mathrm{Im}(\beta\alpha_y - \alpha\beta_y)] = 0.$$

Thus the statement, '$h = 0$', is equivalent to TS 'rule (a)' and this latter rule has been given a meaningful interpretation for all δ.

In a future paper (Cosgrove 1977d), I shall prove that TS 'rule (a)' uniquely determines the solutions of this paper with $h = 0$. Indeed, the more general equation, $(\partial/\partial\nu)h = 0$, interpreting h as the right-hand side of (6.8) or (6.13), if taken as the only constraint on u and ω over and above the stationary axisymmetric vacuum Einstein equations, uniquely determines our three-parameter family of astrophysical solutions but allows a larger class of unphysical solutions.

Throughout the remainder of this paper (except briefly in § 11), I restrict attention to the astrophysical solutions where $h = 0$, $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (1, 0, 0, 1)$ and (3.2) applies.

7. The 'F' equation; return to spheroidal coordinates

In § 4, the Ernst potentials were expressed in terms of the solutions of the second-order K equation. An alternative procedure is to obtain F_1, F_2 and F_3 directly as linearly independent solutions of a third-order linear DE. This equation, though less mathematically simple than the K equation, has some distinct advantages, especially with regard to the investigation of the asymptotically flat outer regions. Later in this section, we shall use the identities (4.8)–(4.10) and (5.9)–(5.11) to construct the F equation in arbitrary curvilinear coordinates and then obtain a compact and very useful form in spheroidal coordinates.

First, consider F_1. From (4.17) and (4.8)–(4.10),

$$F_1 = \Lambda I_0^{-1}(1 + M_0^2), \qquad F_{1\nu} = 2(A - B)\Lambda I_0^{-1}M_0,$$

$$[(2A - 2B)^{-1}F_{1\nu}]_\nu = \Lambda I_0^{-1}(A - BM_0^2).$$

Hence, eliminating M_0 and I_0, and dropping the subscript from F_1, there results the Appell equation (Appell 1889, Cosgrove 1977a):

$$\left(F_{\nu\nu} - \frac{A_\nu - B_\nu}{A - B}F_\nu - (A - B)^2F\right)^2 + (A + B)^2[F_\nu^2 - (A - B)^2F^2] = 0. \tag{7.1}$$

$F = F_1$ is a particular solution; $F = F_3$ is another. The general solution takes the form,

$$F = a^2F_1 + 2abF_2 + b^2F_3, \tag{7.2}$$

where a and b are arbitrary functions of η. However, a look at the η dependence of F, using (5.9)–(5.11), shows that a and b are actually constants (cf (4.3)). All Appell equations may be converted to second-order linear equations (Cosgrove 1977a) or to third-order linear equations (Appell 1889). The former approach leads back to the K equation, the latter to the 'F' equation about to be derived.

If the left-hand side of (7.1) is divided throughout by $(A^2 - B^2)^2$ and then differentiated, the resulting expression factorises into two linear factors, one of the second order and one of the third order. The vanishing of the second-order factor gives rise to two 'singular integrals' which in this case are spurious. The other factor yields a third-order linear Fuchsian differential equation which has F_1, F_2 and F_3 as linearly independent solutions. This equation reads

$$F_{\nu\nu\nu}+\left(\frac{2\eta\nu}{1+\eta\nu^2}-\frac{6\nu}{1-\nu^2}-\frac{2\nu}{\nu^2-e}\right)F_{\nu\nu}+\left(\frac{4\nu^2+d}{(1-\nu^2)(\nu^2-e)}\right.$$
$$\left.-\frac{2\eta\nu^2}{(1+\eta\nu^2)(\nu^2-e)}+\frac{6\nu^2-2-4\delta^2}{(1-\nu^2)^2}+\frac{4H_4-6\eta\nu^2}{(1-\nu^2)(1+\eta\nu^2)}+\frac{\eta}{(1+\eta\nu^2)^2}\right)F_\nu$$
$$+\frac{4(1+\eta)}{(1+\eta\nu^2)(1-\nu^2)^2}\left[-3\eta^2\sigma_1^2\nu+(\sigma_2^2+\eta^2\sigma_1^2\nu^2)\left(\frac{\eta\nu}{1+\eta\nu^2}+\frac{2\nu}{\nu^2-e}\right)\right]F=0, \tag{7.3}$$

where

$$d=d(\eta)=\frac{2\eta^2\sigma_1^2H_2-8H_1H_2^2}{H_1+\eta^2\sigma_1^2H_2}, \qquad e=e(\eta)=\frac{H_1-\sigma_2^2H_2}{H_1+\eta^2\sigma_1^2H_2}. \tag{7.4}$$

This equation has six regular singular points. This number can be reduced to five by changing independent variable to

$$\mu=\nu^2. \tag{7.5}$$

The resulting equation is

$$F_{\mu\mu\mu}+\left(\frac{3}{2\mu}+\frac{\eta}{1+\eta\mu}-\frac{3}{1-\mu}-\frac{1}{\mu-e}\right)F_{\mu\mu}+\frac{1}{4\mu}\left(\frac{4\mu-4(\eta\sigma_1)^{-1}\sigma_2H_2}{(1-\mu)(\mu-e)}\right.$$
$$\left.-\frac{2\eta\mu}{(1+\eta\mu)(\mu-e)}+\frac{4(1-\delta^2)}{(1-\mu)^2}-\frac{10}{1-\mu}+\frac{4H_4+2\eta-8\eta\mu}{(1-\mu)(1+\eta\mu)}+\frac{\eta}{(1+\eta\mu)^2}\right)F_\mu$$
$$+\frac{1+\eta}{2\mu(1+\eta\mu)(1-\mu)^2}\left[-3\eta^2\sigma_1^2+(\sigma_2^2+\eta^2\sigma_1^2\mu)\left(\frac{\eta}{1+\eta\mu}+\frac{2}{\mu-e}\right)\right]F=0. \tag{7.6}$$

These equations are discussed briefly in § 10. A most important property is that the polynomial, $\sigma_2^2+\eta^2\sigma_1^2\nu^2$, does not occur in the denominator of any of the coefficients in (7.3) or (7.6). Thus equations (7.3) and (7.6) are free of the directional 'R singularity' (see the paragraph containing equations (4.21) and (4.22)), though the Appell equation (7.1) is not.

The boundary conditions at $\mu=\nu^2=0$ which distinguish the particular solutions, F_1, F_2 and F_3, are most conveniently expressed in the following manner:

$$\Lambda^{-1}F_1=1+[2(1+\eta)\sigma_2^2+(1+\eta)^{1/2}(\eta\sigma_1-2\sigma_2H_2)]\nu^2+O(\nu^4), \tag{7.7a}$$

$$F_2=2(1+\eta)^{1/2}\sigma_2\nu+O(\nu^3), \tag{7.7b}$$

$$\Lambda F_3=1+[2(1+\eta)\sigma_2^2-(1+\eta)^{1/2}(\eta\sigma_1-2\sigma_2H_2)]\nu^2+O(\nu^4), \tag{7.7c}$$

as $\nu\to 0$. F_1 and F_3 are even in ν; F_2 is odd in ν.

So far, in the K and F equations, η has been held constant. However, analogous ordinary differential equations can be constructed with η as independent variable and ν held constant. They are second order or third order and linear as for the original DE, but they cannot be put in Fuchsian form ($\delta\neq$ integer) since the coefficients are complicated transcendental functions of η, i.e. depending on H_4, etc. Their derivation proceeds from the identities (5.9)–(5.11). But the two cases, ν variable, η constant and η variable, ν constant, are not exhaustive. The identities, (4.8)–(4.10) and (5.9)–(5.11), permit any function of both ν and η to be chosen as independent variable and any function of both ν and η to be held constant.

Now the DE with η held constant are hopelessly inadequate for a discussion of the symmetry axis and its neighbourhood. A limiting procedure in these (ν, η) coordinates would be most awkward. A far better procedure is to abandon the (ν, η) coordinates and choose well behaved coordinates such as spheroidal coordinates (x, y), even though the simple Fuchsian character of the equations is destroyed.

Let us derive the F equation in arbitrary curvilinear coordinates (ρ, τ) defined by

$$\rho=\rho(\nu, \eta), \quad \tau=\tau(\nu, \eta); \qquad \nu=\nu(\rho, \tau), \quad \eta=\eta(\rho, \tau), \tag{7.8}$$

with ρ as independent variable and τ held constant, and show that it is free of the 'R singularity'. Later, we shall see that the equation takes an unexpectedly compact form in both cases of spheroidal coordinates, namely $\rho=x$, $\tau=y$ and $\rho=y$, $\tau=x$.

Start with the identities (4.8)–(4.10) and (5.9)–(5.11). Since

$$\left(\frac{\partial}{\partial\rho}\right)_{\tau\ \text{constant}}=\nu_\rho\frac{\partial}{\partial\nu}+\eta_\rho\frac{\partial}{\partial\eta},$$

these identities imply

$$M_{0\rho}=X+YM_0+ZM_0^2, \tag{7.9a}$$

$$(\Lambda^{-1}I_0)_\rho=(Y+2ZM_0)(\Lambda^{-1}I_0), \tag{7.9b}$$

$$(\Lambda^{-1}J_0)_\rho=-Z(\Lambda^{-1}I_0), \tag{7.9c}$$

where

$$X=\nu_\rho A+\eta_\rho C, \qquad Y=\eta_\rho D, \qquad Z=\nu_\rho B+\eta_\rho E. \tag{7.10}$$

The formulae below will be more compact with the slight change of notation:

$$\begin{aligned} A_1&=A-B, & C_1&=C-E, & X_1&=X-Z,\\ B_1&=A+B, & E_1&=C+E, & Z_1&=X+Z. \end{aligned} \tag{7.11}$$

Now equations (7.9) allow the immediate construction of an Appell equation for F:

$$[\Theta F_{\rho\rho}-\tfrac{1}{2}\Theta_\rho F_\rho-\Theta^2F]^2+\Phi^2[F_\rho^2-\Theta F^2]=0, \tag{7.12}$$

where

$$\Theta=X_1^2+Y^2, \tag{7.13}$$

$$\Phi=Z_1(X_1^2+Y^2)+YX_{1\rho}-X_1Y_\rho. \tag{7.14}$$

On dividing (7.12) throughout by $\Theta\Phi^2$, differentiating and then factoring out the singular integral, there results the third-order linear equation:

$$F_{\rho\rho\rho}-\frac{\Phi_\rho}{\Phi}F_{\rho\rho}+\left(\frac{1}{4}\frac{\Theta_\rho^2}{\Theta^2}-\frac{1}{2}\frac{\Theta_{\rho\rho}}{\Theta}-\Theta+\frac{1}{2}\frac{\Theta_\rho\Phi_\rho}{\Theta\Phi}+\frac{\Phi^2}{\Theta^2}\right)F_\rho+\left(-\frac{3}{2}\Theta_\rho+\Theta\frac{\Phi_\rho}{\Phi}\right)F=0. \tag{7.15}$$

This is the required F equation. However, it is by no means obvious that the 'R singular' terms cancel out of this equation. The proof is as follows. First, the coefficient of $F_{\rho\rho}$ is well behaved if Φ neither vanishes nor is infinite when $R=0$. Now, from (7.14), (7.10) and (7.11),

$$\begin{aligned}\Phi = {} & (\eta_\rho \nu_{\rho\rho} - \nu_\rho \eta_{\rho\rho}) A_1 D + \nu_\rho^3 A_1^2 B_1 \\ & + \nu_\rho^2 \eta_\rho (A_1^2 E_1 + 2A_1 B_1 C_1 + DA_{1\nu} - A_1 D_\nu) \\ & + \nu_\rho \eta_\rho^2 [B_1(C_1^2 + D^2) + DC_{1\nu} - C_1 D_\nu + 2A_1 C_1 E_1 + DA_{1\eta} - A_1 D_\eta] \\ & + \eta_\rho^3 [E_1 (C_1^2 + D^2) + DC_{1\eta} - C_1 D_\eta]. \end{aligned} \tag{7.16}$$

By direct calculation from (4.11)–(4.13), (5.3)–(5.5) and (5.7), we obtain

$$A_1 D = 4\eta^{-1}(1-\nu^2)^{-1} H_1, \tag{7.17a}$$

$$A_1^2 B_1 = 8(1+\eta)(1-\nu^2)^{-3}(1+\eta\nu^2)^{-1}[(1-\nu^2)H_1 - (\sigma_2^2 + \eta^2\sigma_1^2\nu^2)H_2], \tag{7.17b}$$

$$\begin{aligned} & A_1^2 E_1 + 2A_1 B_1 C_1 + DA_{1\nu} - A_1 D_\nu \\ & \quad = 4\eta^{-1}\nu(1-\nu^2)^{-2}(1+\eta\nu^2)^{-1}[3H_2H_4 + 2(1+\eta\nu^2)H_1], \end{aligned} \tag{7.17c}$$

$$\begin{aligned} & B_1(C_1^2 + D^2) + DC_{1\nu} - C_1 D_\nu \\ & \quad = 2(1+\eta)^{-1}(1-\nu^2)^{-1}(1+\eta\nu^2)^{-1}[\eta^{-1}(1-\nu^2)H_1 - (\sigma_1^2 + \sigma_2^2\nu^2)H_2], \end{aligned} \tag{7.17d}$$

$$\begin{aligned} & 2A_1 C_1 E_1 + DA_{1\eta} - A_1 D_\eta \\ & \quad = 2\eta^{-2}(1+\eta)^{-1}(1-\nu^2)^{-1}(1+\eta\nu^2)^{-1}[-(1-\eta\nu^2)H_2H_4 \\ & \qquad + 2(1+2\eta+\eta^2\nu^2)H_1], \end{aligned} \tag{7.17e}$$

$$\begin{aligned} & E_1(C_1^2 + D^2) + DC_{1\eta} - C_1 D_\eta \\ & \quad = \eta^{-2}(1+\eta)^{-2}\nu(1+\eta\nu^2)^{-1}[-H_2H_4 + 2(1+\eta)H_1]. \end{aligned} \tag{7.17f}$$

Clearly, all of the terms in R^{-1} and R^{-2} have cancelled out and, in general, Φ will not vanish when $R=0$. Second,

$$\begin{aligned}\Theta & = \nu_\rho^2 A_1^2 + 2\nu_\rho \eta_\rho A_1 C_1 + \eta_\rho^2 (C_1^2 + D^2) \\ & = \nu_\rho^2 \frac{4R}{(1-\nu^2)^2} - 2\nu_\rho\eta_\rho \frac{2\nu H_4}{\eta(1-\nu^2)(1+\eta\nu^2)} + \eta_\rho^2 \frac{\sigma_1^2 + \sigma_2^2\nu^2}{(1+\eta)(1+\eta\nu^2)}. \end{aligned} \tag{7.18}$$

So the terms in R^{-1} cancel out of Θ also and hence the coefficient of F in (7.15) is also well behaved. However, Θ may vanish when $R=0$, e.g. the cases $\eta_\rho = 0$ such as equations (7.3) or (7.6), and also the case of canonical cylindrical coordinates, $\rho = z$, $\tau = r$. But the coefficient of F_ρ in (7.15) is not singular at any of the zeros of Θ since Θ^2 is a factor of the numerator (taking $\Theta^2\Phi$ as the common denominator). By (7.13) and (7.14) alone, the coefficient of F_ρ may be simplified to

coefficient of F_ρ

$$\begin{aligned} & = -\Theta + \Phi^{-1}[Z_1^3(X_1^2 + Y^2) + 3Z_1^2(YX_{1\rho} - X_1 Y_\rho) + Z_{1\rho}(X_1 X_{1\rho} + YY_\rho) \\ & \quad - Z_1(X_1 X_{1\rho\rho} + YY_{\rho\rho}) + 2Z_1(X_{1\rho}^2 + Y_\rho^2) + (Y_\rho X_{1\rho\rho} - X_{1\rho} Y_{\rho\rho})]. \end{aligned} \tag{7.19}$$

The appropriate boundary conditions for F depend on the curves $\tau(\nu, \eta) =$ constant in the (ν, η) plane. If these curves cut the equatorial plane $\nu = 0$, then the boundary conditions may be given there. They are obtained in a straightforward manner from (7.9), (4.17), (4.18), (4.16) and (4.1). If the curves $\tau =$ constant pass out to the asymptotically flat regions, then the boundary conditions may be given there. The simple asymptotic behaviour of F_1, F_2 and F_3 is given in § 8. It is also possible to express boundary conditions at the infinite red-shift surfaces, $\eta=\eta_0$, $\eta=\eta_1,\ldots$, with known functions.

In general, the $\tau=$ constant curves cut the infinite red-shift surfaces, $\eta=\eta_0$, etc, at isolated points in the (ν,η) plane (exception: $\tau=\eta$). If $\rho=\rho_0=\rho_0(\tau)$ is such a point, then it is a trivial regular singular point of the DE (7.15) with exponents, -1, 0 and 1. It is trivial because the DE for $(\rho-\rho_0)F$ has $\rho=\rho_0$ as an ordinary point.

Finally, for spheroidal coordinates, the functions Θ and Φ take remarkably compact forms. If x is the independent variable while y is held constant, then

$$\Theta=4(1-y^2)^{-2}\sigma_1^2, \tag{7.20}$$

$$\Phi=8y(x^2-y^2)^{-1}(1-y^2)^{-2}H_2\sigma_1^2. \tag{7.21}$$

If y is the independent variable while x is held constant, then

$$\Theta=4(1-y^2)^{-2}\sigma_2^2, \tag{7.22}$$

$$\Phi=-8x(x^2-y^2)^{-1}(1-y^2)^{-2}H_2\sigma_2^2. \tag{7.23}$$

The argument of H_2, σ_1 and σ_2 is, of course, $\eta=(x^2-1)/(1-y^2)$. These formulae will be most useful in the next two sections. It is worth noting also that Θ and Φ take forms nearly as compact as these in cylindrical coordinates.

8. Proof of asymptotic flatness. Physical identification of parameters, m, q and δ

With the F equation of the last section in spheroidal coordinates, it is now easy to determine the behaviour of our solutions in the asymptotically flat outer regions and near the symmetry axis. Since both regions involve large η, it is useful to look at the asymptotic forms of the functions, H_4, H_2, σ_1, σ_2 and Λ, for large η. From (3.1), (3.2) and (3.3)–(3.6),

$$H_4=\delta^2p^{-2}[1+\delta^2q^2p^{-2}\eta^{-1}+\tfrac{1}{2}\delta^2q^2p^{-4}(-p^2+\delta^2p^2+2\delta^2q^2)\eta^{-2}+\ldots], \tag{8.1}$$

$$H_2=\delta qp^{-1}[1+\delta^2p^{-2}\eta^{-1}+\tfrac{1}{4}\delta^2p^{-4}(-p^2+\delta^2p^2+4\delta^2q^2)\eta^{-2}+\ldots], \tag{8.2}$$

$$\sigma_1=\delta p^{-1}\eta^{-1}[1+\delta^2q^2p^{-2}\eta^{-1}+\tfrac{1}{4}\delta^2q^2p^{-4}(-3p^2+3\delta^2p^2+4\delta^2q^2)\eta^{-2}+\ldots], \tag{8.3}$$

$$\sigma_2=\delta^2qp^{-2}\eta^{-1}[1+\tfrac{1}{2}p^{-2}(-p^2+\delta^2p^2+2\delta^2q^2)\eta^{-1}+\ldots], \tag{8.4}$$

$$\Lambda=1-2\delta p^{-1}\eta^{-1/2}+2\delta^2p^{-2}\eta^{-1}+\tfrac{1}{3}\delta p^{-3}(p^2-4\delta^2p^2-6\delta^2q^2)\eta^{-3/2}$$
$$-\tfrac{2}{3}\delta^2p^{-4}(p^2-2\delta^2q^2)\eta^{-2}+\ldots. \tag{8.5}$$

These power series in η^{-1} or $\eta^{-1/2}$ all converge for $\eta>\eta_0$.

Consider the third-order F equation with y as independent variable and x held constant. This is equation (7.15) with $(\rho,\tau)=(y,x)$ and Θ and Φ given by (7.22) and (7.23). Now the points where the $x=$ constant curves $(x^2\neq 1)$ cut the symmetry axis, $y^2=1$, are ordinary points of this DE. This is obvious since

$$\Theta\to 4\delta^4q^2p^{-4}(x^2-1)^{-2}, \qquad \Phi\to -8\delta^5q^3p^{-5}x(x^2-1)^{-3}, \tag{8.6}$$

as $y^2 \to 1$. The boundary conditions on the equatorial plane, $y=0$, are

$$F_1 = \Lambda + \Lambda(2\sigma_2^2 - 2\sigma_2 H_2 x^{-1})y^2 + O(y^4), \quad (8.7a)$$

$$F_2 = 2\sigma_2 y + O(y^3), \quad (8.7b)$$

$$F_3 = \Lambda^{-1} + \Lambda^{-1}(2\sigma_2^2 + 2\sigma_2 H_2 x^{-1})y^2 + O(y^4), \quad (8.7c)$$

where the argument of Λ, σ_2 and H_2 is not $(x^2-1)/(1-y^2)$ but x^2-1.

Now take x large and neglect terms of $O(x^{-4})$ or less. The F equation simplifies drastically. It becomes

$$F_{yyy} + [2p^{-2}y(-2p^2+2\delta^2p^2+3\delta^2q^2)F_{yy} + p^{-2}(-p^2+\delta^2p^2+6\delta^2q^2)F_y]x^{-2} + O(x^{-4}) = 0. \quad (8.8)$$

The three solutions are

$$F_1 = 1 + 2\delta p^{-1}x^{-1} + 2\delta^2p^{-2}x^{-2} + \tfrac{2}{3}\delta p^{-3}[p^2+2\delta^2p^2+3\delta^2q^2(1-y^2)]x^{-3} + O(x^{-4}), \quad (8.9a)$$

$$F_2 = 2\delta^2 q p^{-2} y x^{-2} + O(x^{-4}), \quad (8.9b)$$

$$F_3 = 1 - 2\delta p^{-1}x^{-1} + 2\delta^2p^{-2}x^{-2} - \tfrac{2}{3}\delta p^{-3}[p^2+2\delta^2p^2+3\delta^2q^2(1-y^2)]x^{-3} + O(x^{-4}). \quad (8.9c)$$

The terms up to $O(x^{-2})$ in these formulae are precisely the boundary conditions at infinity for the F equation with x variable, y constant. Clearly, F_1, F_2 and F_3 exhibit the correct asymptotic behaviour for a solution of Ernst's (and hence Einstein's) equations which is asymptotically flat and has symmetry axis free of the ubiquitous line singularity.

From (8.9), (4.1) and (2.3), asymptotic forms may be written down for u, ψ and ω. Convert to the spherical-like coordinates (ρ, θ) defined by (2.7). Also, replace κ by m according to

$$\kappa = mp\delta^{-1}. \quad (8.10)$$

The results are:

$$u = -m\rho^{-1} + \tfrac{1}{3}m^3\delta^{-2}(2\delta^2q^2 - p^2)(\tfrac{3}{2}\cos^2\theta - \tfrac{1}{2})\rho^{-3} + \ldots, \quad (8.11)$$

$$\psi = -2m^2 q \cos\theta(\rho^{-2} - 2m\rho^{-3} + \ldots), \quad (8.12)$$

$$\omega = -2m^2 q \sin^2\theta(\rho^{-1} + m\rho^{-2} + \ldots). \quad (8.13)$$

Comparing with equations (2.10), we discover the following physical identifications for the parameters, m, q and δ:

mass of source $= m$, (8.14a)

angular momentum, $J = m^2 q$, (8.14b)

quadrupole, $Q = m^3[q^2 + \tfrac{1}{3}p^2(1 - 1/\delta^2)]$. (8.14c)

(8.10) and (8.14) are in direct agreement with TS (1973). (At this stage, the additive constant $2\kappa\delta q p^{-1}$ in formula (4.19), which occurs also in (4.20) and (6.5), may be verified by considering equation (4.20) or (6.5) for large x or ρ.)

9. Simple closed formulae on symmetry axis

We know already that the solution is well behaved on the symmetry axis because F_1, F_2 and F_3 are analytic functions of y at $y=\pm 1$, for $x>1$, and have the correct asymptotic forms. But the metric and Ernst potentials actually take very simple functional forms on the symmetry axis.

Consider the F equation with x as independent variable and y constant. The equation is (7.15) with $(\rho, \tau)=(x, y)$ and Θ and Φ given by (7.20) and (7.21). The boundary conditions at $x=\infty$ are given by equations (8.9).

Next, restrict attention to the symmetry axis. There y is constant, in fact $y=\pm 1$, and x varies along it. As $y \to \pm 1$,

$$\Theta \to 4\delta^2 p^{-2}(x^2-1)^{-2}, \qquad \Phi \to \pm 8\delta^3 q p^{-3}(x^2-1)^{-3}. \tag{9.1}$$

Hence the F equation simplifies to

$$F_{xxx}+\frac{6x}{x^2-1}F_{xx}+\frac{6x^2-2-4\delta^2}{(x^2-1)^2}F_x=0. \tag{9.2}$$

This is of hypergeometric type but may be integrated with elementary functions since $x=\infty$ is an ordinary point. The solutions satisfying the boundary conditions (8.9) with $y=\pm 1$ are

$$F_1=\frac{1}{2p^2}\left[(1-p)\left(\frac{x-1}{x+1}\right)^\delta+(1+p)\left(\frac{x+1}{x-1}\right)^\delta-2q^2\right], \tag{9.3}$$

$$F_2=\pm\frac{q}{2p^2}\left[\left(\frac{x-1}{x+1}\right)^\delta+\left(\frac{x+1}{x-1}\right)^\delta-2\right], \tag{9.4}$$

$$F_3=\frac{1}{2p^2}\left[(1+p)\left(\frac{x-1}{x+1}\right)^\delta+(1-p)\left(\frac{x+1}{x-1}\right)^\delta-2q^2\right]. \tag{9.5}$$

The formulae (9.3)–(9.5) and many similar formulae take more convenient forms using the coordinate

$$\zeta=\ln[(1+\nu)/(1-\nu)]=\ln[(x+y)/(x-y)]$$

which when $y=\pm 1$ becomes

$$\zeta=\pm\ln[(x+1)/(x-1)].$$

Considering $y=+1$ only, exact expressions for the metric and Ernst potentials and some other functions and some of their normal derivatives on the symmetry axis are:

$$e^{2u}=p^2(\cosh\delta\zeta+p\sinh\delta\zeta-q^2)^{-1}, \tag{9.6}$$

$$\psi=-2q\sinh^2\tfrac{1}{2}\delta\zeta(\cosh\delta\zeta+p\sinh\delta\zeta-q^2)^{-1}, \tag{9.7}$$

$$\mathscr{E}=(p^2-2iq\sinh^2\tfrac{1}{2}\delta\zeta)(\cosh\delta\zeta+p\sinh\delta\zeta-q^2)^{-1} \tag{9.8a}$$

$$=(p^2+2iq\sinh^2\tfrac{1}{2}\delta\zeta)^{-1}(\cosh\delta\zeta-p\sinh\delta\zeta-q^2), \tag{9.8b}$$

$$\xi=p\coth\tfrac{1}{2}\delta\zeta-iq, \tag{9.9}$$

$$\lim_{y\to 1}(1-y^2)^{-1}\omega=-\kappa\delta q p^{-3}(\cosh\delta\zeta+p\sinh\delta\zeta-1), \tag{9.10}$$

$$u_y = \delta^2 q^2 p^{-2} \sinh^2 \tfrac{1}{2}\zeta \frac{\cosh \delta\zeta + p \sinh \delta\zeta - 1}{\cosh \delta\zeta + p \sinh \delta\zeta - q^2}, \tag{9.11}$$

$$\psi_y = -2\delta^2 pq \sinh^2 \tfrac{1}{2}\zeta \frac{\sinh \delta\zeta + p \cosh \delta\zeta}{(\cosh \delta\zeta + p \sinh \delta\zeta - q^2)^2}, \tag{9.12}$$

$$e^{2\gamma} = 1, \tag{9.13}$$

$$M_0 = \frac{p}{1+q} \frac{\sinh \delta\zeta + p \cosh \delta\zeta}{\cosh \delta\zeta + p \sinh \delta\zeta - q} = \frac{p}{1+q} \frac{q\, e^{\delta\zeta} + 1 - p}{q\, e^{\delta\zeta} - 1 + p}, \tag{9.14}$$

$$I_0 = \frac{2(1-q)}{\cosh \delta\zeta + p \sinh \delta\zeta - q} = \frac{4(1-p)(1-q)\, e^{\delta\zeta}}{(q\, e^{\delta\zeta} - 1 + p)^2} \tag{9.15}$$

$$J_0 = \frac{q \cosh \delta\zeta - 1}{\cosh \delta\zeta + p \sinh \delta\zeta - q} = \frac{q}{1+p} \frac{q\, e^{\delta\zeta} - 1 - p}{q\, e^{\delta\zeta} - 1 + p}, \tag{9.16}$$

$$K_1 = \cosh \tfrac{1}{2}\delta\zeta + p^{-1}(1 - \epsilon i q) \sinh \tfrac{1}{2}\delta\zeta, \tag{9.17}$$

$$K_2 = \epsilon i \cosh \tfrac{1}{2}\delta\zeta - \epsilon i p^{-1}(1 + \epsilon i q) \sinh \tfrac{1}{2}\delta\zeta. \tag{9.18}$$

($\partial/\partial y$ implies that x is held constant.)

These and many other similar formulae are very easy to derive. For $\delta = 1, 2, 3, 4$, they are in agreement with the results of Tomimatsu and Sato (1973). Formula (9.14) with $\delta = 1$ and formulae (9.17) and (9.18) with $\delta = 1, 2, 3$ also agree, respectively, with (4.22) and the tabulated forms of K_1 and K_2 in the appendix when $y = 1$. The corresponding formulae for non-zero NUT parameter are just as easily derived.

10. Series solutions for the H_4 and K equations

In this section, we shall investigate methods of solving the two differential equations, introduced in § 3, for H_4 and K, and discuss some of the properties of their solutions. Since for δ not an integer, the solutions involved are unfamiliar transcendental functions, we consider series solutions whose convergence is quite rapid, even in the highly curved inner regions not too close to the singular surface $\eta = 0$ (or $x = 1$). These methods are applied in § 11 to the cases when δ is an integer to give very efficient methods of calculating the rational function solutions exactly.

10.1. The H_4 and Γ equations

The second-order second-degree equation (3.1) for H_4 is analogous to an Appell equation. If (3.1) is differentiated, a factor H_4'' giving rise to three singular integrals can be removed leaving the third-order first-degree equation:

$$\eta^2(1+\eta)^2 H_4''' + \eta(1+\eta)(1+2\eta)H_4'' + 6\eta(1+\eta)H_4'^2$$
$$- 4(1+2\eta)H_4 H_4' + 2H_4^2 + 4\delta^2 \eta H_4' - 2\delta^2 H_4 = 0. \tag{10.1}$$

This equation, though still non-linear, has many advantages over (3.1). (Note: the singular integrals of (3.1) and (3.14) give rise to a class of static Weyl metrics which contains the Zipoy–Voorhees metrics.) The boundary condition (3.2), when applied to (10.1), still uniquely determines the one-parameter family (for fixed δ) of solutions.

One method of solving equation (10.1), though not the most powerful, is to seek power series of the form,

$$H_4=\delta^2p^{-2}+a_1\eta^{-1}+a_2\eta^{-2}+\dots, \tag{10.2a}$$

or

$$H_4=\delta^2p^{-2}+b_1(1+\eta)^{-1}+b_2(1+\eta)^{-2}+\dots. \tag{10.2b}$$

The method is straightforward, so details will not be given. The series (10.2a) and (10.2b) converge outside circles, centres $\eta=0$ and $\eta=-1$, respectively, in the complex η plane. These circles are the smallest such that H_4 is analytic everywhere outside them and must have at least one singularity on their boundaries. It is shown at the end of this subsection that the only singularities of H_4 are branch-points at $\eta=0$ and $\eta=-1$, requiring the η plane to be cut from $\eta=0$ to $\eta=-1$, and simple poles corresponding to the zeros of Γ. In general, the circle of convergence has radius unity or passes through the real simple pole $\eta=\eta_0$, representing the outermost infinite red-shift surface. In the latter case, the rate of convergence is comparable to a geometric series with common ratio η_0/η or $(1+\eta_0)/(1+\eta)$, respectively. Similar series expansions may be constructed for the function Γ, singular only at $\eta=0$ and $\eta=-1$, using the DE (10.5), below. Though the recurrence relations for the coefficients are qualitatively similar to those for H_4, they are more cumbersome.

A second method of solving (3.1), which we shall describe in detail, is much more powerful and elegant. It is a perturbation expansion of the function $\Gamma(\eta)$ in ascending powers of q^2p^{-2} which is by no means restricted to small q and is valid in the highly curved regions not too close to the singular surfaces $\eta=0$ or $\eta=-1$. Two advantages in the use of $\Gamma(\eta)$ rather than $H_4(\eta)$ are that Γ is analytic for all η except $\eta=0$ and $\eta=-1$ where it has branch-point singularities ($\delta\neq$ integer) and reduces to a polynomial in $1/\eta$ when δ is an integer.

From the definition (3.7) for Γ,

$$H_4=\delta^2+\eta(1+\eta)\Gamma'/\Gamma. \tag{10.3}$$

The two DE for H_4, (3.1) and (10.1), may now be converted to homogeneous DE for Γ, one third order and second degree, the other fourth order and first degree. They are:

$$\begin{aligned}\eta^2(1+\eta)^2(\Gamma^2\Gamma'''^2-6\Gamma\Gamma'\Gamma''\Gamma'''+4\Gamma'^3\Gamma'''-3\Gamma'^2\Gamma''^2+4\Gamma\Gamma''^3)&\\ +4\eta(1+\eta)(1+2\eta)(\Gamma^2\Gamma''\Gamma'''-\Gamma\Gamma'\Gamma''^2+\Gamma'^3\Gamma''-\Gamma\Gamma'^2\Gamma''')&\\ +4(1+2\eta)^2(\Gamma^2\Gamma''^2-\Gamma\Gamma'^2\Gamma'')+4\eta(1+\eta)(\Gamma^2\Gamma'\Gamma'''-2\Gamma\Gamma'^2\Gamma''+\Gamma'^4)&\\ +4(1+2\eta)(2\Gamma^2\Gamma'\Gamma''-\Gamma\Gamma'^3)+4\Gamma^2\Gamma'^2-4\delta^2[(1+\eta)(\Gamma^2\Gamma''^2-2\Gamma\Gamma'^2\Gamma''+\Gamma'^4)&\\ +(2\eta^{-1}+3)(\Gamma^2\Gamma'\Gamma''-\Gamma\Gamma'^3)+\eta^{-2}(1+2\eta)\Gamma^2\Gamma'^2]=0,&\end{aligned} \tag{10.4}$$

$$\begin{aligned}\eta^2(1+\eta)^2(\Gamma\Gamma^{(\mathrm{iv})}-4\Gamma'\Gamma'''+3\Gamma''^2)+4\eta(1+\eta)(1+2\eta)(\Gamma\Gamma'''-\Gamma'\Gamma'')&\\ +[2+14\eta+14\eta^2-4\delta^2(1+\eta)]\Gamma\Gamma''+[-4\eta(1+\eta)+4\delta^2(1+\eta)]\Gamma'^2&\\ +[2+4\eta-\delta^2(4\eta^{-1}+6)]\Gamma\Gamma'=0.&\end{aligned} \tag{10.5}$$

Very similar DE can be written down for $\eta^{\delta^2}\Gamma$. The boundary condition for Γ at $\eta=\infty$ is

$$\Gamma(\eta)=1-\delta^2q^2p^{-2}\eta^{-1}+O(\eta^{-2}) \qquad \text{as } \eta\to\infty. \tag{10.6}$$

We shall regard Γ as a function of two arguments, η and q^2, and write

$$\Gamma = \Gamma(\eta, q^2).$$

Now express Γ in the form of a power series in q^2p^{-2}:

$$\Gamma(\eta, q^2) = \Gamma_0(\eta) + q^2p^{-2}\Gamma_1(\eta) + q^4p^{-4}\Gamma_2(\eta) + \dots \tag{10.7}$$

We shall find that all the coefficients, $\Gamma_0, \Gamma_1, \Gamma_2, \dots$, can be expressed in terms of known functions and quadratures, though the recurrence relation is rather complicated.

Consideration of the series (10.2) shows that

$$\Gamma_0(\eta) = 1 \qquad \text{(exactly)}, \tag{10.8a}$$

$$\Gamma_1(\eta) = -\delta^2\eta^{-1} + O(\eta^{-2}), \tag{10.8b}$$

$$\Gamma_n(\eta) = O(\eta^{-2}), \quad \text{(at least)}, \qquad n \geq 2. \tag{10.8c}$$

These conditions uniquely determine the coefficients in (10.7). Now, differential equations for $\Gamma_1(\eta)$ may be obtained by substituting the series (10.7) into (10.4) and (10.5) and looking at the coefficients of q^4p^{-4} in (10.4) and q^2p^{-2} in (10.5). Actually, the general solution of these DE, not just that one obeying (10.8*b*), is most important. The result will therefore be expressed as follows:

$$v = \Gamma_1'(\eta) \tag{10.9}$$

is a particular solution of both of the DE,

$$[\eta(1+\eta)v'' + 2(1+2\eta)v' + 2v]^2 \\ = 4\delta^2[(1+\eta)v'^2 + (2\eta^{-1}+3)vv' + \eta^{-2}(1+2\eta)v^2], \tag{10.10}$$

$$\eta^2(1+\eta)^2v''' + 4\eta(1+\eta)(1+2\eta)v'' + [2+14\eta+14\eta^2 - 4\delta^2(1+\eta)]v' \\ + [2+4\eta - \delta^2(4\eta^{-1}+6)]v = 0. \tag{10.11}$$

(10.10) is an Appell (1889) equation and (10.11) is a linear Fuchsian equation but not of the ${}_3F_2$ hypergeometric type. If (10.10) is multiplied throughout by $\eta^2(1+\eta)^2$ and then differentiated, a singular integral may be factored out leaving the linear equation (10.11).

The three linearly independent solutions of (10.11) are the foundation stones upon which the whole series (10.7) is constructed. To solve (10.11), convert the Appell equation (10.10) into a second-order linear equation (for the general algorithm, see Cosgrove 1977a). Thus

$$v = \delta^2\eta(1+\eta)w'^2 - \delta^4\eta^{-1}w^2 \tag{10.12}$$

where

$$w'' + \frac{1+2\eta}{\eta(1+\eta)}w' - \frac{\delta^2}{\eta^2(1+\eta)}w = 0. \tag{10.13}$$

The boundary condition (10.8*b*) requires

$$w = \eta^{-1} + O(\eta^{-2}) \qquad \text{as } \eta \to \infty. \tag{10.14}$$

Now (10.13) is of hypergeometric type. It has general solution,

$$w = c_1 W + c_2 W^* \tag{10.15}$$

where

$$W = \eta^{-1}\,{}_2F_1(1+\delta, 1-\delta; 2; -\eta^{-1}) \tag{10.16a}$$

$$= \tfrac{1}{2}\delta^{-1}(\tilde{P}_\delta(1+2\eta^{-1}) - \tilde{P}_{\delta-1}(1+2\eta^{-1})), \tag{10.16b}$$

$$W^* = -\delta(Q_\delta(1+2\eta^{-1}) - Q_{\delta-1}(1+2\eta^{-1})). \tag{10.17}$$

The functions, $\tilde{P}_\delta(\mu)$ and $Q_\delta(\mu)$, are Legendre functions of order δ. $Q_\delta(\mu)$ is the Legendre function of the second kind. $\tilde{P}_\delta(\mu)$ is the Legendre function given by Murphy's formula and is uniquely defined by requiring it to be analytic at $\mu = 1$ and $\tilde{P}_\delta(1) = 1$. If δ is an integer, this is the usual Legendre polynomial, $P_\delta(\mu)$. If δ is not an integer, it is related to the Legendre function of the first kind, $P_\delta(\mu)$, as usually defined, by

$$\tilde{P}_\delta(\mu) = P_\delta(\mu) + \pi^{-1}\tan(\delta\pi)Q_\delta(\mu).$$

The particular solution, $w = W$, corresponds to the boundary condition (10.14). If δ is an integer, this is a polynomial in η^{-1} of degree δ. If δ is not an integer, it is analytic (simple zero) at $\eta = \infty$, but has logarithmic branch-points at $\eta = 0$ and $\eta = -1$. W^* has a logarithmic branch-point at $\eta = \infty$. W and W^* satisfy the important identity,

$$WW^{*\prime} - W^*W' = \eta^{-1}(1+\eta)^{-1}. \tag{10.18}$$

From (10.12) and (10.15), the general solution of (10.10) is

$$v = c_1^2 V + 2c_1c_2 V^* + c_2^2 V^{**} \tag{10.19}$$

where

$$V = \delta^2\eta(1+\eta)W'^2 - \delta^4\eta^{-1}W^2, \tag{10.20}$$

$$V^* = \delta^2\eta(1+\eta)W'W^{*\prime} - \delta^4\eta^{-1}WW^*, \tag{10.21a}$$

$$V^{**} = \delta^2\eta(1+\eta)W^{*\prime 2} - \delta^4\eta^{-1}W^{*2}. \tag{10.21b}$$

These are the three linearly independent solutions of (10.11). $\Gamma_1(\eta)$ is given explicitly by

$$\Gamma_1(\eta) = -\int_\eta^\infty V(\tilde{\eta})\,d\tilde{\eta} \tag{10.22a}$$

$$= \tfrac{1}{2}\delta\int_1^{1+2\eta^{-1}} (\tilde{P}_\delta(\mu)\tilde{P}'_{\delta-1}(\mu) - \tilde{P}_{\delta-1}(\mu)\tilde{P}'_\delta(\mu))\,d\mu. \tag{10.22b}$$

This quadrature cannot be expressed as a simple quadratic function of W and W' or $\tilde{P}_\delta$ and $\tilde{P}'_\delta$. If δ is an integer, $\Gamma_1(\eta)$ is a polynomial in η^{-1} of degree $2\delta - 1$.

The problem of finding $\Gamma_2(\eta), \Gamma_3(\eta), \ldots$ is more complicated. We shall outline the general algorithm for finding $\Gamma_n(\eta)$ in terms of $\Gamma_1, \Gamma_2, \ldots, \Gamma_{n-1}$. The recurrence formula involves only quadratures and will be expressed in a form free of the unwanted Legendre function of the second kind or, equivalently, W^*. As an illustration, $\Gamma_2(\eta)$ will be obtained explicitly.

Substitute the series (10.7) into (10.4) and (10.5). The coefficients of q^6p^{-6} in (10.4) and of q^4p^{-4} in (10.5) give the following inhomogeneous DE for Γ_2:

$$\begin{aligned}\mathscr{M}[\Gamma_2'] \equiv {} & [\eta^2(1+\eta)^2V''+2\eta(1+\eta)(1+2\eta)V'+2\eta(1+\eta)V]\Gamma_2'''\\ & +[2\eta(1+\eta)(1+2\eta)V''+4(1+2\eta)^2V'-4\delta^2(1+\eta)V'+4(1+2\eta)V\\ & -2\delta^2(2\eta^{-1}+3)V]\Gamma_2''+[2\eta(1+\eta)V''+4(1+2\eta)V'\\ & -2\delta^2(2\eta^{-1}+3)V'+4V-4\delta^2\eta^{-2}(1+2\eta)V]\Gamma_2'\\ = {} & \eta^2(1+\eta)^2[3VV'V''-2V'^3]+2\eta(1+\eta)(1+2\eta)[VV'^2+V^2V'']\\ & +[2+12\eta+12\eta^2-4\delta^2(1+\eta)]V^2V'+[2(1+2\eta)-2\delta^2(2\eta^{-1}+3)]V^3,\end{aligned} \tag{10.23}$$

$$\begin{aligned}\mathscr{N}[\Gamma_2'] \equiv {} & \eta^2(1+\eta)^2\Gamma_2^{(\mathrm{iv})}+4\eta(1+\eta)(1+2\eta)\Gamma_2'''+[2+14\eta+14\eta^2-4\delta^2(1+\eta)]\Gamma_2''\\ & +[2+4\eta-\delta^2(4\eta^{-1}+6)]\Gamma_2'\\ = {} & \eta^2(1+\eta)^2[4VV''-3V'^2]+4\eta(1+\eta)(1+2\eta)VV'\\ & +[4\eta(1+\eta)-4\delta^2(1+\eta)]V^2.\end{aligned} \tag{10.24}$$

(10.24) is an inhomogeneous version of the DE (10.11) which has solutions, V, V^* and V^{**}. (10.23) is a linear equation of the second order for Γ_2' and must be a particular first integral of (10.24).

Similarly, the differential equations for Γ_n take the form,

$$\mathscr{M}[\Gamma_n'] = G_n(\eta), \tag{10.25}$$

$$\mathscr{N}[\Gamma_n'] = F_n(\eta). \tag{10.26}$$

$F_n(\eta)$ is a homogeneous quadratic polynomial in $\Gamma_1, \Gamma_2, \ldots, \Gamma_{n-1}$ and their derivatives up to the fourth order. $G_n(\eta)$ is a quartic ($n \geqslant 3$) polynomial in $\Gamma_1, \Gamma_2, \ldots, \Gamma_{n-1}$ and their derivatives up to the third order. For $n \geqslant 3$, F_n and G_n are not expressible directly as polynomials in V and its derivatives, but must involve the quadrature sign. (10.8*c*) expresses the boundary condition for Γ_n. In actual fact, however, Γ_n starts with the power η^{-n^2}.

We shall not solve equations (10.25) and (10.26) by the standard variation of parameters algorithm. Instead, we shall give integrating factors so that the order of the DE can be reduced by taking quadratures. This has the advantage of avoiding the appearance of V^* and V^{**} (or, equivalently, W^* or Q_δ) in the final formulae. First, note the relationship between (10.25) and (10.26):

$$\begin{aligned}(\mathrm{d}/\mathrm{d}\eta)[\eta^2(1+\eta)^2G_n(\eta)]\\ = [\eta^2(1+\eta)^2V''+2\eta(1+\eta)(1+2\eta)V'+2\eta(1+\eta)V]F_n(\eta).\end{aligned} \tag{10.27}$$

If G_n and F_n are replaced by $\mathscr{M}[\Gamma_n']$ and $\mathscr{N}[\Gamma_n']$, respectively, then (10.27) is identically satisfied by Γ_n'. Now let

$$\begin{aligned}\Delta \equiv {} & VV^{*\prime}-V^*V' = \delta^6\eta^{-1}(1+\eta)^{-1}WW'\\ = {} & -\tfrac{1}{2}\delta^2\eta^{-1}(1+\eta)^{-1}[\eta(1+\eta)V''+2(1+2\eta)V'+2V],\end{aligned} \tag{10.28}$$

where (10.10), (10.11), (10.13) and (10.18) have been used. Now $\eta^2(1+\eta)^2\Delta$ is an integrating factor for the DE (10.26). Hence, making use of the relationship (10.27) and identifying the constant of integration by (10.8*c*), we obtain the first integral of (10.26) in the form

$$\Delta\Gamma_n''' - \Delta'\Gamma_n'' + \left(\Delta'' + \frac{4(1+2\eta)}{\eta(1+\eta)}\Delta' + \frac{2+14\eta+14\eta^2-4\delta^2(1+\eta)}{\eta^2(1+\eta)^2}\Delta\right)\Gamma_n'$$

$$= -\frac{\delta^2}{2\eta^2(1+\eta)^2}G_n(\eta). \qquad (10.29)$$

Now the homogeneous equation $\mathcal{M}[v]=0$ has linearly independent solutions, $v=V$ and $v=V^*$ but not $v=V^{**}$. So an integrating factor for (10.29) is $\Delta^{-2}V$. Hence, taking the integral with appropriate constant of integration yields

$$V\Gamma_n'' - V'\Gamma_n' = \tfrac{1}{2}\delta^2\Delta\int_\eta^\infty \tilde{\eta}^{-2}(1+\tilde{\eta})^{-2}\Delta^{-2}(\tilde{\eta})V(\tilde{\eta})G_n(\tilde{\eta})\,d\tilde{\eta}. \qquad (10.30)$$

It is now straightforward to obtain Γ_n explicitly by two more quadratures.

As an illustration of this formula, consider the case $n=2$. $G_2(\eta)$ is the right-hand side of equation (10.23). In terms of W, it is

$$\begin{aligned}G_2(\eta) = {} & 2\delta^{14}\eta^{-3}W^6 + 2\delta^{12}\eta^{-2}(1-\eta)W^5W' - 2\delta^{10}[1+\delta^2(1+\eta^{-1})]W^4W'^2 \\ & -2\delta^{10}(1+\eta)(2+\eta)W^3W'^3 + 2\delta^8\eta(1+\eta)^2(\eta-\delta^2)W^2W'^4 \\ & +2\delta^8\eta^2(1+\eta)^2(1+2\eta)WW'^5 + 2\delta^8\eta^3(1+\eta)^3W'^6. \end{aligned} \qquad (10.31)$$

The integrand of (10.30) involves fractions but these can easily be removed by integrating by parts. Remaining in the integrand is a homogeneous quartic polynomial in W and W'. It is not possible to express the integral of this as a quartic polynomial of the same type, but it can be simplified to a single term by integrating by parts with the aid of (10.13). The result is

$$\begin{aligned}V\Gamma_2'' - V'\Gamma_2' = {} & V^3 - \delta^8W^4W'^2 - \delta^6\eta^2(1+\eta)^2W^2W'^4 \\ & -4\delta^8\eta^{-1}(1+\eta)^{-1}WW'\int_\eta^\infty \tilde{\eta}(1+\tilde{\eta})(2+\tilde{\eta})[W(\tilde{\eta})W'(\tilde{\eta})]^2\,d\tilde{\eta}. \end{aligned} \qquad (10.32)$$

There is no point in carrying out the next two quadratures as the final formulae do not simplify. If δ is an integer, $\Gamma_2(\eta)$ is a polynomial in η^{-1} of degree $4\delta-4$ with lowest power η^{-4}.

Now, (10.30) is a very complicated recurrence relation for the coefficients Γ_n of the series (10.7). For $n>2$ and δ not a small positive integer, calculation of Γ_n is extremely laborious. But, for η not too small, convergence is extremely rapid. This follows from

$$\Gamma_n(\eta) = k_n\eta^{-n^2} + O(\eta^{-n^2-1}) \qquad \text{as } \eta\to\infty, \qquad (10.33)$$

where the coefficient k_n is given by the quite remarkable formula:

$$k_n = (-1)^n\left(\frac{\delta^n(\delta^2-1)^{n-1}(\delta^2-4)^{n-2}(\delta^2-9)^{n-3}\ldots[\delta^2-(n-1)^2]}{n^n(n^2-1)^{n-1}(n^2-4)^{n-2}(n^2-9)^{n-3}\ldots[n^2-(n-1)^2]}\right)^2. \qquad (10.34)$$

For $n \gg \delta$, k_n has asymptotic formula:

$$|k_n| \sim \phi(\delta)(\sin\delta\pi)^{2n-2[\delta]}n^{(2\delta^2-\frac{1}{2})}(2\pi)^{2n}2^{-4n^2} \qquad (10.35)$$

where $\phi(\delta)$ depends on δ only.

Actually, the series (10.7) converges for all finite real and complex values of q^2p^{-2} uniformly with respect to η in the complex η plane with an open neighbourhood of the line segment from $\eta=-1$ to $\eta=0$ removed. Since (10.7) is a power series in q^2p^{-2}, then convergence for $q^2p^{-2}=-1$ implies absolute and uniform (with respect to η) convergence for all complex values of q^2p^{-2} in the circle $|q^2p^{-2}|<1$. The case $q^2p^{-2}=-1$ (i.e. the limit $q^2\to\infty$) may be summed explicitly:

$$1-\Gamma_1(\eta)+\Gamma_2(\eta)-\Gamma_3(\eta)+\ldots\equiv(1+1/\eta)^{\delta^2}. \tag{10.36}$$

This interesting formula is a particular case of the very useful identity,

$$\Gamma(\eta, q^2)\equiv(1+1/\eta)^{\delta^2}\Gamma(-1-\eta, 1/q^2), \tag{10.37}$$

which is easily proved by direct substitution into (10.4) or (10.5) and the boundary condition (10.6). But now, since $\Gamma(\eta, q^2)$ is an analytic function of q^2p^{-2} for $|q^2p^{-2}|<1$ and η not on the cut from -1 to 0, then (10.37) shows that the region of analyticity may be moved to the circle $|q^2p^{-2}+1|<1$. But then $\Gamma(\eta, q^2)$ must be an analytic function of q^2p^{-2} for $|q^2p^{-2}|<2$ because of the form of (10.7). Repeating this argument, it is seen that $\Gamma(\eta, q^2)$ is an analytic function of q^2p^{-2} for all finite complex values of q^2p^{-2} and an analytic function of η in the complex η plane (including $\eta=\infty$) cut from -1 to 0. Note also that (10.37) is useful in the calculation of Γ for large q.

10.2. The K, L and F equations

In this subsection, η is treated as a constant parameter. We shall assume that the values of H_4, H_2, σ_1 and σ_2 for the particular η under consideration are known.

The differential equation (3.8) for $K=K^{(\epsilon)}(\nu, \eta)$ is a linear Fuchsian equation with five regular singular points. They occur at $\nu=1, -1, \mathrm{i}\eta^{-1/2}, -\mathrm{i}\eta^{-1/2}$ and $-\epsilon\mathrm{i}\sigma_2(\eta\sigma_1)^{-1}$. $\nu=\infty$ is an ordinary point. Thus it is one step more complicated than Heun's equation (Heun 1889, Snow 1952, Sleeman 1969). The Riemann P-diagram for this equation is

$$K=P\left\{\begin{matrix} 1 & -1 & \mathrm{i}\eta^{-1/2} & -\mathrm{i}\eta^{-1/2} & -\epsilon\mathrm{i}\sigma_2(\eta\sigma_1)^{-1} & \\ -\frac{1}{2}\delta & -\frac{1}{2}\delta & 0 & 0 & 0 & \nu \\ \frac{1}{2}\delta & \frac{1}{2}\delta & \frac{1}{2} & \frac{1}{2} & 2 & \end{matrix}\right\}. \tag{10.38}$$

In the classification of Ince (1927), this equation is of type [2, 3, 0], meaning it has two elementary singularities (exponent difference $=\frac{1}{2}$), three regular singularities (exponent difference $\neq\frac{1}{2}$) and no irregular singularities.

The fifth regular singular point is interesting. The exponent difference is 2, an integer, but it is easy to show that both linearly independent solutions are free of logarithms at that point. Such a regular singularity is known as an 'apparent' singularity (see Ince 1927). Any Heun equation with an 'apparent' singularity may have that singularity removed so that the general solution is expressible in terms of hypergeometric functions (see Cosgrove 1977b). However, there does not appear to be a similar algorithm which will reduce a Fuchsian equation with five regular singularities, one 'apparent', and one arbitrary accessory parameter, such as (3.8), to a Heun equation.

Solution of equation (3.8) is accomplished by the well known method of power series. It is not necessary to give details of this method here because it may be found in any textbook on differential equations. Power series in $\nu-\nu_0$ involve five-term recurrence relations if ν_0 is one of the regular singular points and six-term recurrence relations otherwise. Further, power series in $(\nu-\nu_0)/(\nu-\nu_1)$ may be constructed and involve four-term, five-term or six-term recurrence relations depending, respectively, on whether both, one or none of ν_0 and ν_1 are singular points. Unfortunately, the most important case, $\nu_0 = 0$ where the boundary conditions (3.9) are directly applicable, is an ordinary point. The regions of convergence in the complex ν plane for such series are the interiors or exteriors of circles and contain no 'non-apparent' singularities apart from ν_0 itself. For example, when $\eta > 1$, the power series in ν converges for $-\eta^{-1/2} <$ (real)$\nu < \eta^{-1/2}$; i.e. convergence is restricted to the region, $0 \leq y^2 < x^2/(2x^2 - 1)$, $x > 1$. This difficulty may be avoided in the upper half space, $y > 0$, by considering power series in $\nu/(\nu - \nu_1)$, where $-2/(\eta - 1) \leq \nu_1 < 0$, and similarly in the lower half space, $y < 0$, with $0 < \nu_1 \leq 2/(\eta - 1)$.

The K equation has several drawbacks. First, power series in ν or $\nu/(\nu - \nu_1)$ require, in general, six-term recurrence relations rather than the optimum four. Second, the region of convergence of any one series does not cover the entire region of interest, $x > 1$, $-1 < y < 1$ (for $y = \pm 1$, see § 9). Third, the singularity at $\nu = -\epsilon i \sigma_2 (\eta \sigma_1)^{-1}$ is a fairly innocuous manifestation of the 'R singularity' but does interfere, to some extent, in discussion of the asymptotically flat outer regions. Fourth, the equation conceals the symmetry property evident in equations (3.11*a*, *b*) and (3.12). This last symmetry property may be exploited to choose a new dependent variable. There is much freedom here, but the following eight choices seem to be optimum. Let $\epsilon = \pm 1$, $\epsilon_2 = \pm 1$, $\epsilon_3 = \pm 1$ independently and write

$$L = L^{(\epsilon, \epsilon_2, \epsilon_3)} = \left(\frac{1 - \epsilon\epsilon_2 i \eta^{1/2} \nu}{1 + \epsilon\epsilon_2 i \eta^{1/2} \nu}\right)^{1/4} K^{(\epsilon)} + \epsilon_3 \left(\frac{1 + \epsilon\epsilon_2 i \eta^{1/2} \nu}{1 - \epsilon\epsilon_2 i \eta^{1/2} \nu}\right)^{1/4} K^{(-\epsilon)}. \tag{10.39}$$

For $K = K_1$ or K_2, L is either an even or an odd function of ν. The DE for L is in independent variable $\mu = \nu^2$ and is

$$L_{\mu\mu} + \left(\frac{1}{2\mu} + \frac{1}{\mu - 1} + \frac{\eta}{1 + \eta\mu} - \frac{1}{\mu - k}\right) L_\mu + \left[-\frac{(1+\eta)(\sigma_2^2 + \eta^2 \sigma_1^2 \mu)}{4\mu (1-\mu)^2 (1+\eta\mu)}\right.$$
$$+ \tfrac{1}{2}\epsilon_3 (1+\eta)^{1/2} \left(\frac{1}{\mu - k} - \frac{1}{2\mu}\right) \frac{\eta\sigma_1 + \epsilon_2 \eta^{1/2} \sigma_2}{(1-\mu)(1+\eta\mu)}$$
$$\left. + \frac{1}{4\mu}\left(\frac{\frac{1}{2}\epsilon_2 \eta^{1/2}}{1 + \eta\mu} + \frac{H_2}{1-\mu}\right)^2\right] L = 0, \tag{10.40}$$

where

$$k = k(\eta) = \frac{\frac{1}{2}\epsilon_2 \eta^{1/2} + H_2 + \epsilon_3 (1+\eta)^{1/2} \sigma_2}{\frac{1}{2}\epsilon_2 \eta^{1/2} - \eta H_2 + \epsilon_2 \epsilon_3 \eta^{3/2} (1+\eta)^{1/2} \sigma_1}. \tag{10.41}$$

This DE is of Ince type [3, 2, 0]. Its Riemann P-diagram is

$$L^{(\epsilon, \epsilon_2, \epsilon_3)} = P \begin{Bmatrix} 0 & \infty & 1 & -\eta^{-1} & k(\eta) & \\ 0 & 0 & -\frac{1}{2}\delta & -\frac{1}{4} & 0 & \mu \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2}\delta & \frac{1}{4} & 2 & \end{Bmatrix}. \tag{10.42}$$

The quadratic transformation, $\mu = \nu^2$, caused elementary singularities to appear at $\mu = 0, \infty$. The regular singularity at $\mu = k$ is again 'apparent'. If this singularity were absent, the DE would be equivalent to Lamé's equation (see, e.g., Arscott 1964). Thus L is qualitatively very similar to a well known special function, Lamé's function, but the recurrence formula for the power series in μ has four terms rather than three. Power series in $\mu/(1 + \eta\mu)$ require the optimum four-term recurrence relations and converge rapidly throughout the region, $x>1$, $-1<y<1$, except near the poles, $x=1$, $y=\pm 1$, of the singular surface, $x=1$. Note also that (10.40) is manifestly free of the 'R singularity'.

The inverse of equation (10.39) is

$$K^{(\epsilon)}=\left(\frac{1+\epsilon\epsilon_2 \mathrm{i}(\eta\mu)^{1/2}}{1-\epsilon\epsilon_2 \mathrm{i}(\eta\mu)^{1/2}}\right)^{1/4}\left[-\epsilon \mathrm{i}\frac{\mu^{1/2}(1-\mu)(1+\eta\mu)}{l(\mu-k)}L_\mu + \frac{1}{2}\left(1+\frac{\epsilon\epsilon_3 \mathrm{i}(1+\eta)^{1/2}(\eta\sigma_1+\epsilon_2\eta^{1/2}\sigma_2)\mu^{1/2}}{l(\mu-k)}\right)L\right] \qquad (10.43)$$

where $l=l(\eta)$ is the denominator of $k(\eta)$ in the expression (10.41). The boundary conditions on L are most conveniently expressed in the following way. If

$$L=a_0+a_1\mu+a_2\mu^2+\ldots, \qquad (10.44a)$$

then, from (10.43),

$$K^{(\epsilon)}=\begin{cases}\frac{1}{2}a_0K_1^{(\epsilon)} & \text{if } \epsilon_3=+1\\ -\frac{1}{2}\epsilon \mathrm{i} a_0K_2^{(\epsilon)} & \text{if } \epsilon_3=-1.\end{cases} \qquad (10.44b)$$

If

$$L=\mu^{1/2}(b_0+b_1\mu+b_2\mu^2+\ldots), \qquad (10.45a)$$

then

$$klK^{(\epsilon)}=\begin{cases}\frac{1}{2}b_0K_2^{(\epsilon)} & \text{if } \epsilon_3=+1\\ \frac{1}{2}\epsilon \mathrm{i} b_0K_1^{(\epsilon)} & \text{if } \epsilon_3=-1.\end{cases} \qquad (10.45b)$$

These results, together with (4.2), allow the Ernst potentials to be calculated.

On the infinite red-shift surfaces, six of the eight $L^{(\epsilon,\epsilon_2,\epsilon_3)}$ reduce to Lamé functions, being related in a very simple way to the function $\bar{K}$ satisfying (4.26). The two cases, $\epsilon=\pm 1$, $\epsilon_2=\epsilon_3=-1$, are different and give rise to different Fuchsian equations, types [2, 4, 0] in ν or [3, 2, 0] in μ.

A disadvantage with the use of L is that the relationship to the Ernst potentials is somewhat remote. Here, the F equation, (7.3) or (7.6), is most attractive. The Riemann P-diagram for (7.6) is

$$F=P\begin{Bmatrix}0 & \infty & 1 & -\eta^{-1} & e(\eta) & \\ 0 & 0 & -\delta & 0 & 0 & \\ \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} & 1 & \mu \\ 1 & 1 & \delta & \frac{3}{2} & 3 & \end{Bmatrix}. \qquad (10.46)$$

$\mu=e$ is an 'apparent' singularity which causes no trouble. Some of the other exponents differ by unity but do not give rise to logarithmic solutions. Power series in μ and $\mu/(1+\eta\mu)$ involve five-term recurrence relations, the latter converging throughout the region, $x>1$, $-1<y<1$, as for the function L.

11. The cases when δ is an integer: Tomimatsu–Sato solutions

The methods of the previous section allow us to construct the rational function cases, δ an integer. The polynomials $\Gamma(\eta, q^2)$ have so many remarkable symmetry properties that they are worthy of study in their own right by pure mathematicians. The symmetry properties are also important because, with their aid, we can greatly reduce the number of applications of the recurrence formula (10.30). Many of the properties of these polynomials are given without proof.

In this section, in preference to $\Gamma(\eta, q^2)$, it is convenient to use

$$\bar{\Gamma}(\eta, q^2) \equiv p^{2\delta}\eta^{\delta^2}\Gamma(\eta, q^2) \tag{11.1}$$

which is a polynomial in η of degree δ^2 and in q^2 of degree δ.

Consider, first, the K or L equation when δ is an integer. A glance at the Riemann P-diagrams, (10.38) and (10.42), shows that the regular singularities at $\mu = \nu^2 = 1$ have exponent difference δ, an integer. But no logarithms occur in the solutions at $\mu = \nu^2 = 1$ (see, for example, the tabulated forms of K_1 and K_2 for $\delta = 1, 2, 3$ in the appendix). In fact, the general solution of (3.8) takes the form:

$$\begin{aligned} K = (1-\nu^2)^{-\frac{1}{2}\delta} & \text{ (polynomial in } \nu \text{ of degree } \delta) \\ & + (1-\nu^2)^{-\frac{1}{2}\delta}(1+\eta\nu^2)^{1/2} \text{ (polynomial in } \nu \text{ of degree } \delta - 1). \end{aligned} \tag{11.2}$$

But the freedom from logarithms condition at $\mu = \nu^2 = 1$ is not satisfied identically by the parameters in (3.8) but leads to a new relation among H_4, H_2, σ_1 and σ_2. This may be re-arranged to form a first-order DE for H_4 whose form depends on δ. This DE is a first integral of the second-order H_4 equation (3.1) and is precisely that unique first integral which admits the boundary condition (3.2).

For $\delta = 1$ (Kerr solution), the condition that the K and L equations are free of logarithms at $\mu = \nu^2 = 1$ may be reduced to

$$\eta(1+\eta)H_4' + H_4(H_4 - 1) = 0. \tag{11.3}$$

This is a Bernoulli equation for H_4 which may be readily integrated to give $H_4 = \eta(\eta p^2 - q^2)^{-1}$. Changing dependent variable to $\bar{\Gamma}$ by (10.3) and (11.1) gives the simple linear equation,

$$\bar{\Gamma}'' = 0. \tag{11.4}$$

For $\delta = 2$, the freedom from logarithms condition is

$$\begin{aligned} \eta^2(1+\eta)^2 H_4'^2 + 2\eta(1+\eta)[H_4^2 + (2\eta - 3)H_4 + 2\eta^2 - 2\eta + 2]H_4' \\ + H_4(H_4 - 4)(H_4 + 2\eta - 1)^2 = 0. \end{aligned} \tag{11.5}$$

In terms of $\bar{\Gamma}$, this equation becomes an Appell equation,

$$\begin{aligned} \eta^2(1+\eta)^2\bar{\Gamma}''^2 - 4\eta(1+\eta)(1+2\eta)\bar{\Gamma}'\bar{\Gamma}'' + 4(3\eta^2 + 3\eta + 1)\bar{\Gamma}\bar{\Gamma}'' + 12\eta(1+\eta)\bar{\Gamma}'^2 \\ - 12(1+2\eta)\bar{\Gamma}\bar{\Gamma}' = 0. \end{aligned} \tag{11.6}$$

On dividing (11.6) throughout by $\eta^2(1+\eta)^2$, then differentiating and factoring out the singular integral, there results

$$\eta(1+\eta)\bar{\Gamma}''' - 2(1+2\eta)\bar{\Gamma}'' + 6\bar{\Gamma}' = 0. \tag{11.7}$$

This equation is of hypergeometric type and is easily solved to give the quartic polynomial tabulated in the appendix (equation (A.2)).

Now write

$$\bar{\Gamma}(\eta, q^2) = p^{2\delta}\bar{\Gamma}_0(\eta) + p^{2\delta-2}q^2\bar{\Gamma}_1(\eta) + p^{2\delta-4}q^4\bar{\Gamma}_2(\eta) + \ldots + p^2 q^{2\delta-2}\bar{\Gamma}_{\delta-1}(\eta) + q^{2\delta}\bar{\Gamma}_\delta(\eta) \tag{11.8}$$

so that

$$\bar{\Gamma}_0(\eta) = \eta^{\delta^2}, \qquad \bar{\Gamma}_\delta(\eta) = (-1)^\delta, \qquad \bar{\Gamma}_n(\eta) = \eta^{\delta^2}\Gamma_n(\eta).$$

Notice that the two linearly independent solutions of (11.4) are $\bar{\Gamma}_0(\eta)$ and $\bar{\Gamma}_1(\eta)$ and that the three linearly independent solutions of (11.7) are $\bar{\Gamma}_0(\eta)$, $\bar{\Gamma}_1(\eta)$ and $\bar{\Gamma}_2(\eta)$. This pattern continues. The freedom from logarithms condition leads to a second-order DE for $\bar{\Gamma}$ which is a generalisation of Appell's equation in the sense that the general solution takes the form (11.8) with p^2 and q^2 regarded as independent. By successive differentiation and removal of singular integrals, a linear equation of order $\delta + 1$ may be obtained. This equation has $\bar{\Gamma}_0(\eta), \bar{\Gamma}_1(\eta), \ldots, \bar{\Gamma}_\delta(\eta)$ as linearly independent solutions. The case $\delta = 3$ is given by

$$\eta^2(1+\eta)^2\bar{\Gamma}^{(\text{iv})} - 8\eta(1+\eta)(1+2\eta)\bar{\Gamma}''' + 14(1+7\eta+7\eta^2)\bar{\Gamma}'' - 112(1+2\eta)\bar{\Gamma}' = 0. \tag{11.9}$$

For all $\delta \geq 2$, these linear equations are of Fuchsian type with exactly three regular singular points occurring at $\eta = 0, -1$ and ∞, but, except for $\delta = 2$, they are not of the generalised hypergeometric type.

The DE (11.4), (11.7), (11.9), etc, may be obtained with much less labour directly from their Riemann P-diagrams because the exponents at the singular points obey an amazingly simple rule. The Riemann P-diagram is

$$\bar{\Gamma}(\eta) = P\begin{Bmatrix} 0 & -1 & \infty & \\ 0 & 0 & -\delta^2 & \\ 1^2 & 1^2 & -\delta^2+1^2 & \\ 2^2 & 2^2 & -\delta^2+2^2 & \eta \\ \vdots & \vdots & \vdots & \\ \delta^2 & \delta^2 & 0 & \end{Bmatrix}. \tag{11.10}$$

This diagram does not quite uniquely determine the linear equation of order $\delta + 1$ for $\bar{\Gamma}$ because of arbitrary accessory parameters (see, e.g., Ince 1927). But more than enough information is available when the highest and lowest powers of η in the polynomials, $\bar{\Gamma}_n(\eta)$, $n = 0, 1, \ldots, \delta$, are known. They follow the very simple pattern:

$$\bar{\Gamma}_n(\eta) = k_n\eta^{\delta^2-n^2} + \ldots + l_n\eta^{(\delta-n)^2}, \tag{11.11}$$

where k_n is given by (10.34) and $l_n = (-1)^\delta k_{\delta-n}$.

A glance at the tabulated forms of $\bar{\Gamma}$ in the appendix for $\delta = 1, 2, 3, 4, 5$ reveals the simple symmetry property,

$$\bar{\Gamma}(\eta, q^2) \equiv (-1)^\delta\eta^{\delta^2}\bar{\Gamma}(1/\eta, p^2). \tag{11.12}$$

This is equivalent to 'rule (c)' of Tomimatsu and Sato (1973). It is, however, true only for the cases when δ is an integer. If δ is not an integer, the right-hand side of (11.12) satisfies the DE for $\bar{\Gamma}$ but not the boundary condition (7.6). Another symmetry property, true for all δ, is given by (10.37). It may be written

$$\bar{\Gamma}(\eta, q^2) \equiv q^{2\delta}\bar{\Gamma}(-1-\eta, 1/q^2). \tag{11.13}$$

(11.12) and (11.13) lead to three more symmetries for the cases when δ is an integer:

$$\bar{\Gamma}(\eta, q^2) \equiv (-1)^\delta p^{2\delta} \eta^{\delta^2} \bar{\Gamma}(-1-1/\eta, 1/p^2), \tag{11.14a}$$

$$\bar{\Gamma}(\eta, q^2) \equiv q^{2\delta}(1+\eta)^{\delta^2} \bar{\Gamma}(-1/(1+\eta), -p^2/q^2), \tag{11.14b}$$

$$\bar{\Gamma}(\eta, q^2) \equiv (-1)^\delta p^{2\delta}(1+\eta)^{\delta^2} \bar{\Gamma}(-\eta/(1+\eta), -q^2/p^2). \tag{11.14c}$$

The last of these is a manifestation of the identity, $P_\delta(-\mu) \equiv (-1)^\delta P_\delta(\mu)$, for Legendre polynomials (recall that $\mu = 1 + 2/\eta$ in equations (10.16*b*) and (10.22*b*)). These symmetries form a finite group of order 6.

These symmetries lead to useful identities among the coefficients of (11.8). (11.12) gives

$$\bar{\Gamma}_n(\eta) \equiv (-1)^\delta \eta^{\delta^2} \bar{\Gamma}_{\delta-n}(1/\eta). \tag{11.15}$$

By expanding both sides of (11.13) and comparing coefficients of powers of $1/p^2$, we obtain a sequence of identities. The first, second and $(n+1)$th of these are

$$(1+\eta)^{\delta^2} \equiv \eta^{\delta^2} - \bar{\Gamma}_1(\eta) + \bar{\Gamma}_2(\eta) - \ldots + (-1)^{\delta-1}\bar{\Gamma}_{\delta-1}(\eta) + 1, \tag{11.16}$$

$$(-1)^\delta \bar{\Gamma}_1(-1-\eta) \equiv -\bar{\Gamma}_1(\eta) + 2\bar{\Gamma}_2(\eta) - 3\bar{\Gamma}_3(\eta) + \ldots + (-1)^{\delta-1}(\delta-1)\bar{\Gamma}_{\delta-1}(\eta) + \delta, \tag{11.17}$$

$$\begin{aligned}(-1)^\delta \bar{\Gamma}_n(-1-\eta) \equiv{}& (-1)^n \bar{\Gamma}_n(\eta) + {}^{n+1}C_n(-1)^{n+1}\bar{\Gamma}_{n+1}(\eta) + {}^{n+2}C_n(-1)^{n+2}\bar{\Gamma}_{n+2}(\eta) + \ldots \\ &+ {}^{\delta-1}C_n(-1)^{\delta-1}\bar{\Gamma}_{\delta-1}(\eta) + {}^\delta C_n,\end{aligned} \tag{11.18}$$

where ${}^rC_n$ is the binomial coefficient $r!/(r-n)!\,n!$.

Now the most efficient method of calculating the polynomials $\bar{\Gamma}_n(\eta)$ in (11.8) makes use of the identities, (11.15) and (11.16)–(11.18), the form (11.11), the explicit formula (10.22) for $\Gamma_1 = \eta^{-\delta^2}\bar{\Gamma}_1$ and the recurrence relation (10.30) used as few times as possible. To calculate $\bar{\Gamma}(\eta, q^2)$ for a particular δ, first decide by trial and error the integer n (roughly, $n \simeq \frac{1}{2}(\sqrt{2}-1)\delta \simeq 0{\cdot}21\delta$) with the following property. Consider the $\delta - 2n - 1$ polynomials, $\bar{\Gamma}_{n+1}, \bar{\Gamma}_{n+2}, \ldots, \bar{\Gamma}_{\delta-n-1}$. (11.11) allows several of these polynomials to contain the same power of η. Let m be the largest number of these polynomials which contain the same power of η. Then make $2n+1 \geqslant m$ with $2n+1$ as close as possible to m. It is sufficient now to compute $\bar{\Gamma}_1, \bar{\Gamma}_2, \ldots, \bar{\Gamma}_n$ from the recurrence relation (10.30). Then only $\bar{\Gamma}_{n+1}, \bar{\Gamma}_{n+2}, \ldots, \bar{\Gamma}_{\frac{1}{2}\delta}$ or $\bar{\Gamma}_{\frac{1}{2}(\delta-1)}$ remain to be determined. Let these be polynomials with undetermined coefficients subject to the restrictions (11.11). Next, substitute into the first $n+1$ of the sequence of identities (11.16)–(11.18). These identities will furnish enough equations to determine all of the unknown coefficients. Indeed, in most cases, some of the coefficients are overdetermined and, if m is even, they all are. So the method also provides a way of checking for errors in the expressions found for $\bar{\Gamma}_1, \ldots, \bar{\Gamma}_n$.

If $\bar{\Gamma}_1(\eta)$ only is computed from (10.22), then $\bar{\Gamma}(\eta, q^2)$ can be fully constructed by the above method for $1 \leqslant \delta \leqslant 7$. If $\bar{\Gamma}_1(\eta)$ and $\bar{\Gamma}_2(\eta)$ are computed from (10.22) and (10.32), then $\bar{\Gamma}(\eta, q^2)$ can be constructed for $1 \leqslant \delta \leqslant 12$. At this stage, the polynomials are becoming quite large—the middle coefficients in $(1+\eta)^{144}$ are of the order of 10^{42}.

Another interesting property of the polynomials $\bar{\Gamma}(\eta, q^2)$ is that the functions H_2, σ_1 and σ_2, whose definitions involve square roots, are rational functions of η with $\bar{\Gamma}$ as denominator. Also $\Lambda(\eta)$ is a rational function of $(1+\eta)^{1/2}$. Splitting Λ into its numerator and denominator,

$$\Lambda = \Sigma_1/\Sigma_2, \tag{11.19}$$

we have the following remarkable identity:

$$\bar{\Gamma} \equiv \Sigma_1 \Sigma_2. \tag{11.20}$$

Thus the zeros of $\bar{\Gamma}$ are either zeros or poles of Λ, as already noted in § 4. The functions, H_4, H_2, σ_1, σ_2, Σ_1 and Σ_2, are tabulated in the appendix for $\delta = 1, 2, 3$. Several more symmetries are evident in these formulae.

To complete the calculation of the full metric and Ernst potentials, once $\bar{\Gamma}$, H_4, H_2, σ_1, σ_2 and Λ are known, is much more straightforward. (11.2) shows the form of the solution of the K equation. The labour is comparable if the F equation is used. The form of the general solution is:

$$F = (1-\nu^2)^{-\delta}(\text{polynomial in } \nu \text{ of degree } 2\delta) \\ + (1-\nu^2)^{-\delta}(1+\eta\nu^2)^{1/2}(\text{polynomial in } \nu \text{ of degree } 2\delta - 1). \tag{11.21}$$

Recall that F_1 and F_3 are even in ν, F_2 is odd in ν.

It is of considerable mathematical interest to note that there are series of non-transcendental solutions even when $h \neq 0$. They occur not for δ an integer but for

$$(\delta^2 + 2h)^{1/2} = \text{integer} \quad \text{and} \quad (\delta^2 - 2h)^{1/2} = \text{integer}. \tag{11.22}$$

We shall not discuss these solutions here but will be content with presenting the solution of the $\bar{\Gamma}$ equation explicitly for the two simplest cases. Let $\epsilon_4 = \pm 1$ and define $\bar{\Gamma} = \eta^{\delta^2}\Gamma$ where Γ is defined by (10.3) and H_4 satisfies (3.14). Then

$$\delta^2 = 1 - 2\epsilon_4 h \Rightarrow \bar{\Gamma} = p^2\eta^{\alpha(\delta,h)} - q^2\eta^{\beta(\delta,h)}, \tag{11.23}$$

$$\delta^2 = 4 - 2\epsilon_4 h \Rightarrow \bar{\Gamma} = p^4\eta^{\alpha(\delta,h)} - p^2q^2\eta^{-\epsilon_4 h}[(4-4\epsilon_4 h)\eta^3 + (6-8\epsilon_4 h)\eta^2 + (4-4\epsilon_4 h)\eta] \\ + q^4\eta^{\beta(\delta,h)}, \tag{11.24}$$

where $\alpha(\delta, h)$ and $\beta(\delta, h)$ are the two roots of the quadratic,

$$x^2 - \delta^2 x + h^2 = 0.$$

These obviously reduce to the formulae (A.1) and (A.2) of the appendix when $h = 0$. Likewise, the K and F equations may also be solved with elementary functions. In addition, the Ricatti equation (5.15) for $\mu(\eta)$ may be solved with elementary functions. However, the final formulae for the metric and Ernst potentials are much less compact than the corresponding formulae for $h = 0$.

12. Conclusions

The three-parameter family of solutions of Einstein's equations described in the preceding pages is exceedingly complicated when δ is not an integer. The metric coefficients and Ernst potentials are constructed from two functions, $H_4(\eta)$ and $K^{(\epsilon)}(\nu, \eta)$, satisfying second-order ordinary differential equations in independent variables η and ν, respectively. The K equation is a linear Fuchsian equation and involves $H_4(\eta)$ and some closely related functions of η as constant parameters. A knowledge of the function $H_4(\eta)$ alone is sufficient to construct all the metric coefficients on the equatorial plane and the metric coefficient $e^{2\gamma}$ everywhere. On the infinite red-shift surfaces, $\eta = \eta_0$, etc, the functions, $K^{(\epsilon)}(\nu, \eta_0)$, etc, are familiar transcendental functions of ν, depending in a simple way on Lamé functions. On the symmetry axis, the metric coefficients and Ernst potentials take very simple functional forms.

It was found necessary to set up some mathematical preliminaries in order to prove that the Ernst potential given in § 4 actually represents an asymptotically flat vacuum gravitational field. With the aid of the set of six identities (4.8)–(4.10) and (5.9)–(5.11), it was then a straightforward process of substitution to show that Einstein's and Ernst's equations and also Tomimatsu–Sato 'rule (a)' were satisfied. Using the same set of identities, we formulated the F equation in spheroidal coordinates and used the two cases to show that the metrics were asymptotically flat and well behaved on the symmetry axis. The series solutions developed (or briefly mentioned if straightforward) in § 10 for the two transcendental ($\delta \neq$ integer) functions, H_4 and K, converge rapidly everywhere outside the singular surface $x = 1$ except near this surface. In § 11, efficient algorithms for constructing the rational Tomimatsu–Sato series of solutions ($\delta =$ integer) were outlined.

A larger class of solutions (six non-trivial parameters) has been shown to arise by introducing a new parameter h into the differential equations and by relaxing the boundary conditions. When $h \neq 0$, a third differential equation was found necessary to construct the full metric. These additional solutions are not astrophysically meaningful.

In a future paper (Cosgrove 1977d), I shall show how the present six-parameter family and another new family of solutions may be derived as a 'simple' solution of a new and rather unusual formulation of the stationary axisymmetric vacuum field equations. This may be compared with Ernst (1968) showing that the Kerr solution takes an extremely simple form in his complex potential formalism. However, this method of derivation is not the method by which these solutions were discovered and is less powerful and elegant than this latter method. These solutions arose from the discovery of a new transformation group which constructs new solutions from old whilst preserving the boundary conditions appropriate to a finite rotating body in empty space. The transformation, unfortunately, does not give rotation to the static Weyl metrics but merely permutes these metrics among themselves. The solutions of this paper are the invariants of the transformation. The derivation of both the group and the invariant solutions will be given in Cosgrove (1977e). Future research should reveal more transformation groups (even, perhaps, a systematic theory of them) and therefore more asymptotically flat solutions which will, without doubt, assume extremely complicated functional forms. The most attractive feature of this theory, apart from its apparent power, is that the astrophysical boundary conditions can be incorporated at an early stage in attempts to construct new solutions. Most of the many and varied mathematical techniques used in solving Einstein's equations in recent times, e.g. algebraic speciality, have very little or no control over the boundary conditions.

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Appendix. Table of $\bar{\Gamma}(\eta, q^2)$ for $\delta = 1, 2, 3, 4, 5$ and $H_4(\eta)$, $H_2(\eta)$, $\sigma_1(\eta)$, $\sigma_2(\eta)$, $\Lambda(\eta)$, $K_1(\nu, \eta)$ and $K_2(\nu, \eta)$ for $\delta = 1, 2, 3$

If δ is an integer, $\bar{\Gamma}(\eta, q^2) = p^{2\delta}\eta^{\delta^2}\Gamma$ is a polynomial in η of degree δ^2 and a homogeneous polynomial in p^2 and q^2 of degree δ. It is very closely related to the metric coefficient $e^{2\gamma}$ (see (4.5) or (6.10)). For values of δ not too large, these polynomials may be calculated relatively easily by the methods of § 11. The first five are:

$$\delta = 1: \qquad \bar{\Gamma} = p^2\eta - q^2; \tag{A.1}$$

$$\delta = 2: \qquad \bar{\Gamma} = p^4\eta^4 - p^2q^2(4\eta^3 + 6\eta^2 + 4\eta) + q^4; \tag{A.2}$$

$\delta = 3$:

$$\bar{\Gamma} = p^6\eta^9 - p^4q^2(9\eta^8 + 36\eta^7 + 84\eta^6 + 90\eta^5 + 36\eta^4) + p^2q^4(36\eta^5 + 90\eta^4 + 84\eta^3 + 36\eta^2 + 9\eta) - q^6; \tag{A.3}$$

$\delta = 4$:

$$\begin{aligned}\bar{\Gamma} = p^8\eta^{16} &- p^6q^2(16\eta^{15} + 120\eta^{14} + 560\eta^{13} + 1420\eta^{12} + 1968\eta^{11} + 1400\eta^{10} + 400\eta^9)\\ &+ p^4q^4(400\eta^{12} + 2400\eta^{11} + 6608\eta^{10} + 11040\eta^9 + 12870\eta^8\\ &+ 11040\eta^7 + 6608\eta^6 + 2400\eta^5 + 400\eta^4) - p^2q^6(400\eta^7 + 1400\eta^6\\ &+ 1968\eta^5 + 1420\eta^4 + 560\eta^3 + 120\eta^2 + 16\eta) + q^8;\end{aligned} \tag{A.4}$$

$\delta = 5$:

$$\begin{aligned}\bar{\Gamma} = p^{10}\eta^{25} &- p^8q^2(25\eta^{24} + 300\eta^{23} + 2300\eta^{22} + 10150\eta^{21} + 26880\eta^{20} + 43400\eta^{19}\\ &+ 41800\eta^{18} + 22050\eta^{17} + 4900\eta^{16}) + p^6q^4(2500\eta^{21} + 26250\eta^{20}\\ &+ 133700\eta^{19} + 438900\eta^{18} + 1059525\eta^{17} + 2007450\eta^{16}\\ &+ 3023760\eta^{15} + 3553200\eta^{14} + 3158400\eta^{13} + 2041900\eta^{12}\\ &+ 904200\eta^{11} + 245000\eta^{10} + 30625\eta^9) - p^4q^6(30625\eta^{16} + 245000\eta^{15}\\ &+ 904200\eta^{14} + 2041900\eta^{13} + 3158400\eta^{12} + 3553200\eta^{11} + 3023760\eta^{10}\\ &+ 2007450\eta^9 + 1059525\eta^8 + 438900\eta^7 + 133700\eta^6 + 26250\eta^5\\ &+ 2500\eta^4) + p^2q^8(4900\eta^9 + 22050\eta^8 + 41800\eta^7 + 43400\eta^6\\ &+ 26880\eta^5 + 10150\eta^4 + 2300\eta^3 + 300\eta^2 + 25\eta) - q^{10}.\end{aligned} \tag{A.5}$$

The functions H_4, H_2, σ_1 and σ_2 are derived from $\bar{\Gamma}$ by (10.3), (3.3), (3.4) and (3.5), respectively. H_1 need not be tabulated since $H_1 = \frac{1}{2}\eta\sigma_1\sigma_2$. For $\delta = 1$,

$$\bar{\Gamma}H_4 = \eta, \tag{A.6}$$

$$\bar{\Gamma}H_2 = pq(1 + \eta), \tag{A.7}$$

$$\bar{\Gamma}\sigma_1 = p, \tag{A.8}$$

$$\bar{\Gamma}\sigma_2 = q. \tag{A.9}$$

For $\delta = 2$,

$$\bar{\Gamma}H_4 = 4\eta[p^2\eta^3 - q^2], \tag{A.10}$$

$$\bar{\Gamma} H_2 = 2pq(1+\eta)[p^2(\eta^3+3\eta^2) - q^2(3\eta+1)], \tag{A.11}$$

$$\bar{\Gamma}\sigma_1 = 2p[p^2\eta^3 + q^2(3\eta+2)], \tag{A.12}$$

$$\bar{\Gamma}\sigma_2 = 2q[p^2(2\eta^3+3\eta^2)+q^2]. \tag{A.13}$$

For $\delta = 3$,

$$\bar{\Gamma} H_4 = 9\eta[p^4\eta^8 - p^2q^2(16\eta^5+30\eta^4+16\eta^3)+q^4], \tag{A.14}$$

$$\bar{\Gamma} H_2 = 3pq(1+\eta)[p^4(\eta^8+8\eta^7+10\eta^6) - p^2q^2(18\eta^6+56\eta^5+70\eta^4+56\eta^3+18\eta^2) + q^4(10\eta^2+8\eta+1)], \tag{A.15}$$

$$\bar{\Gamma}\sigma_1 = 3p[p^4\eta^8 + p^2q^2(18\eta^6+52\eta^5+60\eta^4+24\eta^3)+q^4(10\eta^2+12\eta+3)], \tag{A.16}$$

$$\bar{\Gamma}\sigma_2 = 3q[p^4(3\eta^8+12\eta^7+10\eta^6)+p^2q^2(24\eta^5+60\eta^4+52\eta^3+18\eta^2)+q^4]. \tag{A.17}$$

The function $\Lambda(\eta)$ is a rational function of $(1+\eta)^{1/2}$ for δ an integer. We shall tabulate the numerator Σ_1 and denominator Σ_2 for $\delta = 1, 2, 3$. They obey

$$\Lambda = \Sigma_1/\Sigma_2, \qquad \bar{\Gamma} = \Sigma_1\Sigma_2. \tag{(A.18)}$$

The real zeros ($\eta > -1$) of Σ_1 and Σ_2 represent the infinite red-shift surfaces but only those represented by the zeros of Σ_1 carry the ring singularity on the equatorial plane.

For $\delta = 1$,

$$\Sigma_1 = p(1+\eta)^{1/2}+1, \tag{A.19}$$

$$\Sigma_2 = p(1+\eta)^{1/2}-1. \tag{A.20}$$

For $\delta = 2$,

$$\Sigma_1 = p^2(\eta^2+2\eta) - q^2 + 2p\eta(1+\eta)^{1/2}, \tag{A.21}$$

$$\Sigma_2 = p^2(\eta^2+2\eta) - q^2 - 2p\eta(1+\eta)^{1/2}. \tag{A.22}$$

For $\delta = 3$,

$$\Sigma_1 = p(1+\eta)^{1/2}[p^2(\eta^4+4\eta^3) - q^2(6\eta^2+4\eta+1)] + p^2(3\eta^4+4\eta^3) - q^2, \tag{A.23}$$

$$\Sigma_2 = p(1+\eta)^{1/2}[p^2(\eta^4+4\eta^3) - q^2(6\eta^2+4\eta+1)] - p^2(3\eta^4+4\eta^3) + q^2. \tag{A.24}$$

The functions, $K_1(\nu, \eta)$ and $K_2(\nu, \eta)$, both take the form (11.2). The coefficients of the polynomials in ν are rational functions of $(1+\eta)^{1/2}$ with Σ_1 and Σ_2, respectively, as common denominator. For $\delta = 1$,

$$K_1 = \Sigma_1^{-1}(1-\nu^2)^{-1/2}[(1+\eta)^{1/2}(p-\epsilon i q\nu) + (1+\eta\nu^2)^{1/2}], \tag{A.25}$$

$$K_2 = \epsilon i \Sigma_2^{-1}(1-\nu^2)^{-1/2}[(1+\eta)^{1/2}(p-\epsilon i q\nu) - (1+\eta\nu^2)^{1/2}]. \tag{A.26}$$

For $\delta = 2$,

$$K_1 = \Sigma_1^{-1}(1-\nu^2)^{-1}[-1+\eta^2\nu^2+(1+\eta)^2(p-\epsilon i q\nu)^2 + 2(1+\eta)^{1/2}(1+\eta\nu^2)^{1/2}(p\eta - \epsilon i q\nu)], \tag{A.27}$$

$$K_2 = \epsilon i \Sigma_2^{-1}(1-\nu^2)^{-1}[-1+\eta^2\nu^2+(1+\eta)^2(p-\epsilon i q\nu)^2 - 2(1+\eta)^{1/2}(1+\eta\nu^2)^{1/2}(p\eta - \epsilon i q\nu)]. \tag{A.28}$$

For $\delta = 3$,

$$K_1 = \Sigma_1^{-1}(1-\nu^2)^{-3/2}[\![(1+\eta)^{1/2}\{-p(6\eta^2+4\eta+1-3\eta^4\nu^2) + \epsilon iq\nu[3-(\eta^4+4\eta^3+6\eta^2)\nu^2] + (1+\eta)^4(p-\epsilon iq\nu)^3\} + (1+\eta\nu^2)^{1/2}\{p^2(3\eta^4+4\eta^3) - q^2 - 12\epsilon ipq\eta(1+\eta)^2\nu + [p^2\eta^4 - q^2(4\eta+3)]\nu^2\}]\!], \quad (A.29)$$

$$K_2 = \epsilon i\Sigma_2^{-1}(1-\nu^2)^{-3/2}[\![(1+\eta)^{1/2}\{-p(6\eta^2+4\eta+1-3\eta^4\nu^2) + \epsilon iq\nu[3-(\eta^4+4\eta^3+6\eta^2)\nu^2] + (1+\eta)^4(p-\epsilon iq\nu)^3\} - (1+\eta\nu^2)^{1/2}\{p^2(3\eta^4+4\eta^3) - q^2 - 12\epsilon ipq\eta(1+\eta)^2\nu + [p^2\eta^4 - q^2(4\eta+3)]\nu^2\}]\!]. \quad (A.30)$$

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